Online Appendix for

The Limits of Political Compromise: Debt Ceilings and Political Turnover

Alexandre B. Cunha
Federal University of Rio de Janeiro

Emanuel Ornelas
Sao Paulo School of Economics-FGV, CEPR, CESifo and CEP

This appendix contains four sections, described as follows:

(I) In the context of example 2, we show how to compute $\bar{B}$, how to construct functions f^b and Γ, and that all assumptions introduced in subsections 4.1.2 and 4.1.3 are satisfied under reasonable parameter restrictions.
(II) We show that the spendthrift policy is an equilibrium outcome.
(III) We study the properties of the policy g^N that maximizes the net gain from cooperation, $NGC(g)$.
(IV) We characterize a set of static equilibrium outcomes and study how it changes with n and B_L.

We follow the numbering scheme of equations, propositions, lemmas, etc. of the main body of the paper.

I The value $\bar{B}$, functions f^b and Γ, and assumptions from subsections 4.1.2 and 4.1.3 under example 2

We show here how to compute $\bar{B}$ and construct the functions f^b and Γ in the context of example 2. We also show that all assumptions introduced in subsections 4.1.2 and 4.1.3 are satisfied in the context of that example provided that the relevant parameters satisfy some reasonable restrictions.

Computing the maximum level of debt $\bar{B}$. Let B_s denote the maximum value that b can assume in a steady state. Clearly, $B_s \leq \bar{B}$. On the other hand, $\bar{B}$ has to be attainable in a steady state; thus, $\bar{B} \leq B_s$. We conclude that $\bar{B} = B_s$. Therefore, to find $\bar{B}$ it is enough to find the maximum attainable steady-state value of b. In such a context, $b = b'$. Hence, (20) becomes

$$b = \frac{1}{1-\beta}(\tau l - g).$$

To evaluate $\bar{B}$, it is enough to select an attainable vector (τ, g) to maximize $(\tau l - g)$. Let us carry out this task for the preferences in example 2. Given that l satisfies (9), we have to set g equal to its minimum value γ. We should set τ equal to the upper bound $\bar{\tau}$. As a consequence,

$$\bar{B} = \frac{1}{1-\beta}\left(\bar{\tau}\frac{a_1}{a_1+a_2} - \gamma\right).$$

Functions f^b and Γ, constraints (25) and (26), and inequality (27). We start with the construction of function f^b. Combine (20) with (9) to obtain

$$b_{t+1} = \beta^{-1}\left(b_t + g_t - \tau_t\frac{a_1}{a_1+a_2}\right) \geq \beta^{-1}\left(b_t + \gamma - \bar{\tau}\frac{a_1}{a_1+a_2}\right).$$

Now, define f^b according to

$$f^b(b_t) \equiv \beta^{-1}\left(b_t + \gamma - \bar{\tau}\frac{a_1}{a_1+a_2}\right). \tag{54}$$

It is a straightforward exercise to show that $f^b(\bar{B}) = \bar{B}$. Moreover, the definition of f^b implies that $b_{t+1} \geq f^b(b_t)$, while the definition of $\bar{B}$ implies that $b_{t+1} \leq \bar{B}$. Therefore, (25) is satisfied.

Consider now Γ and (26). Again, combine (20) with (9). This procedure leads to

$$g_t = \tau_t\frac{a_1}{a_1+a_2} + \beta b_{t+1} - b_t \leq \bar{\tau}\frac{a_1}{a_1+a_2} + \beta b_{t+1} - b_t.$$

Therefore,

$$\Gamma(b_t, b_{t+1}) \equiv \bar{\tau}\frac{a_1}{a_1+a_2} + \beta b_{t+1} - b_t. \tag{55}$$

Note that $\Gamma(\bar{B}, \bar{B}) = \gamma$. Since $g_t \geq \gamma$ and $g_t \leq \Gamma(b_t, b_{t+1})$, (26) holds.

Given that $\Gamma_b = -1$ and $\Gamma_{b'} = \beta$, $\Gamma_b(b,b) + \Gamma_{b'}(b,b) = -1 + \beta < 0$. Hence, inequality (27) is satisfied.

Inequalities (30), (31), and (32). As pointed out in the discussion of example 2,

$$a_1 - (a_1 + a_2)(g + b - \beta b') > 0.$$

Now, differentiate (37) and use the above inequality to conclude that

$$U_b(b, g, b') = -\frac{a_1(a_1 + a_2)}{a_1 - (a_1 + a_2)(g + b - \beta b')} < 0, \tag{56}$$

$$U_g(b, g, b') = -\frac{a_1(a_1 + a_2)}{a_1 - (a_1 + a_2)(g + b - \beta b')} + \frac{a_3}{g}, \tag{57}$$

$$U_{b'}(b, g, b') = \frac{\beta a_1(a_1 + a_2)}{a_1 - (a_1 + a_2)(g + b - \beta b')} > 0, \tag{58}$$

$$U_{gg}(b, g, b') = -\left\{\frac{a_1(a_1 + a_2)^2}{[a_1 - (a_1 + a_2)(g + b - \beta b')]^2} + \frac{a_3}{g^2}\right\} < 0, \tag{59}$$

$$U_{bg}(b, g, b') = -\frac{a_1(a_1 + a_2)^2}{[a_1 - (a_1 + a_2)(g + b - \beta b')]^2} < 0, \tag{60}$$

$$U_{gb'}(b, g, b') = \frac{\beta a_1(a_1 + a_2)^2}{[a_1 - (a_1 + a_2)(g + b - \beta b')]^2} > 0, \tag{61}$$

and

$$U_{bg}(b, g, b) + U_{gb'}(b, g, b) = -\frac{(1 - \beta)a_1(a_1 + a_2)^2}{\{a_1 - (a_1 + a_2)[g + (1 - \beta)b]\}^2} < 0.$$

Hence, all inequalities in (30), (31) and (32) are satisfied in the context of example 2.

Condition (33). Let b and $\hat{b}$ be two attainable debt levels satisfying $b < \hat{b}$. Combine (56) with (58) to conclude that $U_b(b, g, b) + U_{b'}(b, g, b) < 0$. Apply this result to conclude that $U(\hat{b}, g^*(\hat{b}, \hat{b}), \hat{b}) < U(b, g^*(\hat{b}, \hat{b}), b)$. On the other hand, $U(b, g^*(\hat{b}, \hat{b}), b) \leq U(b, g^*(b, b), b)$. Combine the last two inequalities to obtain the desired result.

Condition (34). First, let us introduce two conditions. Inequalities

$$\bar{\tau} > \frac{a_3}{a_1 + a_3} \tag{62}$$

and

$$\gamma < (1 - \bar{\tau})\frac{a_3}{a_1 + a_2} \tag{63}$$

ensure that the maximum tax rate $\bar{\tau}$ is not too small and that γ is not too large.

Clearly, there is nothing to show if $g^*(b, b') = \Gamma(b, b')$. Hence, assume that $g^*(b, b') < \Gamma(b, b')$. Let $g^u(b, b')$ be the unconstrained maximizer of $U(b, g, b')$. If $g^u(b, b') > \gamma$, then $g^*(b, b') = g^u(b, b')$ and we have the desired result. Thus, we conclude by showing that if $g^u(b, b') \leq \gamma$, then (63) is violated.

By setting the partial derivative in (57) equal to zero, we conclude that

$$g^u(b, b') = \frac{a_3}{a_1 + a_3}\left(\frac{a_1}{a_1 + a_2} + \beta b' - b\right). \tag{64}$$

Therefore,

$$\gamma \geq g^u(b,b') \Rightarrow \gamma \geq \frac{a_3}{a_1+a_3}\frac{a_1}{a_1+a_2} + \frac{a_3}{a_1+a_3}(\beta b' - b). \tag{65}$$

Now, combine the fact that $b' \geq f^b(b)$ and (54) to conclude that

$$\beta b' - b \geq \gamma - \bar{\tau}\frac{a_1}{a_1+a_2}.$$

Together with (65), the last inequality implies that

$$\gamma \geq \frac{a_3}{a_1+a_3}\frac{a_1}{a_1+a_2} + \frac{a_3}{a_1+a_3}\left(\gamma - \bar{\tau}\frac{a_1}{a_1+a_2}\right).$$

Reorganizing this inequality, we obtain

$$\gamma \geq (1-\bar{\tau})\frac{a_3}{a_1+a_2}.$$

Inequality (35). Combine (55), (62), and (64) to conclude that

$$g^u(0,0) = \frac{a_3}{a_1+a_3}\frac{a_1}{a_1+a_2} < \bar{\tau}\frac{a_1}{a_1+a_2} = \Gamma(0,0).$$

Thus, it remains to show that $g^*(0,0) = g^u(0,0)$. Given that $g^u(0,0) < \Gamma(0,0)$, the last equality will hold if $g^u(0,0) > \gamma$. Now, observe that (62) implies that

$$1 - \bar{\tau} < 1 - \frac{a_3}{a_1+a_3} = \frac{a_1}{a_1+a_3}.$$

Combine this result with (63). This yields

$$\gamma < \frac{a_1}{a_1+a_3}\frac{a_3}{a_1+a_2} = g^u(0,0).$$

II The spendthrift equilibrium

We state in the main body of the paper that the spendthrift policy is a symmetric political equilibrium if conditions $(C1)$ and $(C2)$ are satisfied. In this section we prove that assertion. We first discuss it in an intuitive way in subsection II.1. A formal argument is presented in subsection II.2. That argument relies on some properties of the partial derivatives of the function G. We show in subsection II.3 that the properties in question are satisfied in the context of example 2.

II.1 The underlying intuition

In this subsection we intuitively discuss why conditions $(C1)$-$(C2)$ ensure that the spendthrift policy is an equilibrium outcome. Since λ measures politicians' degree of profligacy, it may appear that condition $(C1)$ alone would be enough to ensure that the spendthrift policy is an equilibrium outcome. However, this need not be true. The reason is that a high λ represents a penchant for rents today but also in the future, and setting $b_1 = \bar{B}$ would decrease future rents to their minimum level.[44] It is then clear that some additional requirement is needed to ensure that the spendthrift policy is an equilibrium outcome. Condition $(C2)$ does just that.

To see the role of condition $(C2)$, take a policy $\{g_t, b_{t+1}\}_{t=s}^{\infty}$ with the property that $g_t = G(b_t, b_{t+1}, \lambda)$. For simplicity, assume that the partial derivatives G_b and $G_{b'}$ are defined at every point (b, b', λ). Let t be any date and δ a small positive number. If b_{t+1} increases by δ, g_t will grow by approximately $\delta G_{b'}(b_t, b_{t+1}, \lambda)$ while g_{t+1} will fall by approximately $-\delta G_b(b_{t+1}, b_{t+2}, \lambda)$. Note that G_b and $G_{b'}$ include possible endogenous changes in taxation due to changes in g_t. Hence, a policymaker can substitute g_t for g_{t+1} at the rate

$$-\frac{\delta G_{b'}(b_t, b_{t+1}, \lambda)}{\delta G_b(b_{t+1}, b_{t+2}, \lambda)} = -\frac{G_{b'}(b_t, b_{t+1}, \lambda)}{G_b(b_{t+1}, b_{t+2}, \lambda)}.$$

In a symmetric outcome, the derivatives of the date-t incumbent's payoff with respect to g_t and g_{t+1} are equal to, respectively, $U_g(b_t, g_t, b_{t+1}) + \lambda$ and $\beta[U_g(b_{t+1}, g_{t+1}, b_{t+2}) + \lambda/n]$. Therefore,

$$-\frac{dg_t}{dg_{t+1}} = \beta \frac{U_g(b_{t+1}, g_{t+1}, b_{t+2}) + \lambda/n}{U_g(b_t, g_t, b_{t+1}) + \lambda},$$

where $-dg_t/dg_{t+1}$ is a standard intertemporal marginal rate of substitution. Thus, the date-t incumbent has an incentive to increase g_t and to reduce g_{t+1} by issuing debt whenever

$$-\frac{G_{b'}(b_t, b_{t+1}, \lambda)}{G_b(b_{t+1}, b_{t+2}, \lambda)} > \beta \frac{U_g(b_{t+1}, g_{t+1}, b_{t+2}) + \lambda/n}{U_g(b_t, g_t, b_{t+1}) + \lambda}.$$

Now, in line with $(C1)$, make $\lambda \to \infty$. Since U_g is bounded, the right-hand side of the inequality above converges to β/n. Hence, for λ sufficiently large, it becomes

$$-\frac{G_{b'}(b_t, b_{t+1}, \lambda)}{G_b(b_{t+1}, b_{t+2}, \lambda)} > \frac{\beta}{n}.$$

For this condition to hold for all n, we need that

$$-\frac{G_{b'}(b_t, b_{t+1}, \lambda)}{G_b(b_{t+1}, b_{t+2}, \lambda)} > \frac{\beta}{2}. \tag{66}$$

Thus, if λ is large, the date-t incumbent will always have an incentive to issue debt and increase g_t if inequality (66) holds, which is precisely what condition $(C2)$ ensures. In the next subsection we provide a more general version of (66) that takes into consideration, among other technical issues, that G_b and $G_{b'}$ may be undefined at some points (b, b', λ).

To better understand the nature of condition (66), consider that the economy is in a steady state, so that $b_t = b_{t+1}$ for all t. In such a context, one can show that the left-hand side of (66) would be equal to β and the inequality would be trivially satisfied. Thus, one can interpret inequality (66) as a condition that ensures that the ratio $G_{b'}(b_t, b_{t+1}, \lambda)/G_b(b_{t+1}, b_{t+2}, \lambda)$ does not deviate too much from its steady-state value.

[44] It is easy to see, for example, that a dictator would set $g_t = g^D$ and $b_{t+1}^D = 0$ for every t, thus keeping the public debt unchanged regardless of the value of λ.

II.2 A formal argument

We establish here that the spendthrift policy is an equilibrium outcome if conditions $(C1)$ and $(C2)$ are satisfied. We proceed in steps, since this is a long exercise. Those steps consist of:

1. Proving that the function G is strictly decreasing in b, strictly increasing in b', and increasing in λ.
2. Showing that there exists a number λ_1 that does not depend on (b, b') with the property that $\lambda > \lambda_1 \Rightarrow G(b, b', \lambda) = \Gamma(b, b')$ for every (b, b').
3. Showing that the partial derivatives of G are bounded.
4. Establishing a technical condition that is similar but more general than the intuitive constraint (66) of the previous subsection, which in turn corresponds to condition $(C2)$.
5. Characterizing the part of the expected payoff of the date-t incumbent that depends on that player's actions and showing that it is strictly increasing in b_{t+1} for every t if $(C1)$ holds (i.e., if λ is sufficiently large).
6. Showing that, under $(C1)$ and $(C2)$, an incumbent will always be willing to increase the public debt until it reaches the upper bound $\bar{B}$, and therefore the spendthrift policy plan constitutes a symmetric political equilibrium.

Let us outline how each of these steps fits into our task. We do that in a reverse order. Consider step 6, which is the last and most important one. It is carried out in Proposition 8. Step 5 consists of establishing Lemma 8, which is used in the proof of Proposition 8. Step 4 consists of spelling out two conditions, namely (72) and (74), on the derivatives of G. They are used in the proof of Lemma 8. Step 3 is formalized in Lemma 7, which in turn is applied to prove Lemma 8. Step 2, used to obtain step 3, consists of establishing Lemma 6. Finally, step 1, employed to carry out the subsequent step, is formalized in Lemma 5.

We need to introduce some notation. We denote the solution of the unconstrained version of (38) by G^u. That is, $G^u(b, b', \lambda)$ is the maximizer of $U(b, g, b') + \lambda g$. Since U is strictly concave in g, condition (34) implies that $U_g(b, \gamma, b') > 0$ if $\Gamma(b, b') > \gamma$. As a consequence, if the last inequality holds, then $G^u(b, b', \lambda) > \gamma$.

Lemma 5 *The function G is strictly decreasing in b, strictly increasing in b', and increasing in λ.*

Proof. Let G^u_b, $G^u_{b'}$, and G^u_λ denote the partial derivatives of G^u. We adopt similar notation for the partial derivatives of G and Γ. The differentiation of (40) when it holds with equality establishes that

$$G^u_b = -\frac{U_{bg}}{U_{gg}},\ G^u_{b'} = -\frac{U_{gb'}}{U_{gg}} \text{ and } G^u_\lambda = -\frac{1}{U_{gg}}. \tag{67}$$

Recall that $U_{gg} < 0$. Therefore, $G^u_\lambda(b, b', \lambda) > 0$. Then, combine the former inequality with (31) to conclude that $G^u_b(b, b', \lambda) < 0$ and $G^u_{b'}(b, b', \lambda) > 0$.

The function G may fail to be differentiable exactly when $G^u(b,b',\lambda)=\Gamma(b,b')$. However, G is differentiable whenever $G^u(b,b',\lambda)\neq\Gamma(b,b')$. Suppose that $G^u(b,b',\lambda)<\Gamma(b,b')$; thus, $G(b,b',\lambda)=G^u(b,b',\lambda)$ and $G_b=G^u_b<0$, $G_{b'}=G^u_{b'}>0$, and $G_\lambda=G^u_\lambda>0$. If $G^u(b,b',\lambda)>\Gamma(b,b')$, then $G(b,b',\lambda)=\Gamma(b,b')$; as a consequence, $G_b=\Gamma_b<0$, $G_{b'}=\Gamma_{b'}>0$, and $G_\lambda=f^g_\lambda=0$.

Let G^-_b and G^+_b denote, respectively, the left and right derivatives of G with respect to b. We use analogous notation for the side derivatives with respect to b' and λ. It should be clear from the previous paragraph that G^-_b is equal to G^u_b or Γ_b. Similarly, $G^+_b=G^u_b$ or $G^+_b=\Gamma_b$. The same reasoning applies to the side derivatives with respect to b' and λ. Therefore,

$$G^-_b<0,\ G^+_b<0,\ G^-_{b'}>0,\ G^+_{b'}>0,\ G^-_\lambda\geq 0, \text{ and } G^+_\lambda\geq 0.$$

Even if G is not differentiable when $G^u(b,b',\lambda)=\Gamma(b,b')$, the inequalities above allow us to conclude that G is strictly decreasing in b, strictly increasing in b' and increasing in λ. Consider the variable b. At a point where G_b is not defined, both the left G^-_b and the right G^+_b partial derivatives are negative. Since G is continuous, we can be sure that its value decreases as b increases. Analogous reasoning applies to b' and λ. ■

Lemma 6 *There exists a number λ_1 that does not depend on (b,b') with the property that, if $\lambda>\lambda_1$, then $G(b,b',\lambda)=\Gamma(b,b')$ for every (b,b').*

Proof. The definition of G^u implies that $U_g(\bar{B},G^u(\bar{B},-\underline{B},\lambda),-\underline{B})=-\lambda$. Therefore, $\lim_{\lambda\to\infty}U_g(\bar{B},G^u(\bar{B},-\underline{B},\lambda),-\underline{B})=-\infty$. Recall the definition of $\bar{\Gamma}$ from the proof of lemma 5 in the appendix of the main text. Since $U_g(\bar{B},\bar{\Gamma},-\underline{B})>-\infty$, then there must exist a number λ_1 with the property that, if $\lambda>\lambda_1$, then

$$\bar{\Gamma}\leq G^u(\bar{B},-\underline{B},\lambda). \tag{68}$$

Now, observe that both b and b' belong to $[-\underline{B},\bar{B}]$. Hence, $b\leq\bar{B}$ and $b'\geq-\underline{B}$. Use the fact that $G^u_b<0$ and $G^u_{b'}>0$ to conclude that

$$G^u(\bar{B},-\underline{B},\lambda)\leq G^u(b,b',\lambda)$$

for every (b,b'). Combine the last inequality with (68) to conclude that if $\lambda>\lambda_1$, then $\bar{\Gamma}\leq G^u(b,b',\lambda)$ for every (b,b'). However, $\Gamma(b,b')\leq\bar{\Gamma}$. Hence, $G^u(b,b',\lambda)\geq\Gamma(b,b')$ whenever $\lambda>\lambda_1$. Thus, $G(b,b',\lambda)=\Gamma(b,b')$ for every $\lambda>\lambda_1$. ■

Our next step consists of showing that some of the partial derivatives of G are bounded. Taking into account that G_b may be undefined at some points, we need to establish that

$$\sup_{(b,b',\lambda)}\left[\max\{|G^-_b(b,b',\lambda)|,|G^+_b(b,b',\lambda)|\}\right]<\infty. \tag{69}$$

Observe that if G_b is defined everywhere, then $G^-_b=G^+_b$ and (69) is equivalent to $\sup_{(b,b',\lambda)}|G_b(b,b',\lambda)|<\infty$. In a similar fashion, we have to prove that

$$\sup_{(b,b',\lambda)}\left[\max\{|G^-_{b'}(b,b',\lambda)|,|G^+_{b'}(b,b',\lambda)|\}\right]<\infty. \tag{70}$$

Lemma 7 *The partial derivatives of G satisfy (69) and (70).*

Proof. Since $(b,b') \in [-\underline{B},\bar{B}] \times [-\underline{B},\bar{B}]$, we can conclude that $\sup_{(b,b')} |\Gamma_b(b,b')| < \infty$. Now, take any λ larger than λ_1. Lemma 6 implies that G_b is well defined and equal to Γ_b. Therefore, (69) holds if we impose the extra condition that $\lambda > \lambda_1$. If $\lambda \leq \lambda_1$, then (b,b',λ) lies in a compact set; hence, $\sup_{(b,b',\lambda)} |G_b^u(b,b',\lambda)| < \infty$. Moreover, G_b^- is equal to G_b^u or Γ_b; the same applies to G_b^+. Thus, both G_b^- and G_b^+ are bounded for $\lambda \leq \lambda_1$. Hence, (69) holds if we impose the extra condition that $\lambda \leq \lambda_1$. Since (69) holds for $\lambda > \lambda_1$ and $\lambda \leq \lambda_1$, it clearly holds if we do not place any constraint on λ. Similar reasoning establishes that (70) holds. ■

We now lay out a technical condition that is equivalent to the intuitive constraint (66) on the partial derivatives of G_b and $G_{b'}$. For a moment, assume that those derivatives are well defined. Use the fact that $G_b < 0$ to rewrite (66) as

$$G_{b'}(b,b',\lambda) - \frac{\beta}{2}|G_b(b',b'',\lambda)| > 0,$$

where b'' denotes the public debt two dates ahead. For technical reasons, we need the left-hand side of that inequality to be bounded away from zero. That is,

$$G_{b'}(b,b',\lambda) - \frac{\beta}{2}|G_b(b',b'',\lambda)| \geq \varepsilon$$

for some positive ε. After we take into consideration that G_b and $G_{b'}$ may be undefined at some points, the last inequality has to be replaced by

$$G_{b'}^-(b,b',\lambda) - \frac{\beta}{2}|G_b^-(b',b'',\lambda)| \geq \varepsilon \tag{71}$$

and

$$G_{b'}^+(b,b',\lambda) - \frac{\beta}{2}|G_b^+(b',b'',\lambda)| \geq \varepsilon. \tag{72}$$

For our purposes, it is possible to replace inequalities (71) and (72) with two much weaker conditions. It suffices to assume that there exist numbers $\tilde{\lambda}_0 > 1$, $\varepsilon > 0$ and $\eta \in [0,1)$ such that, if $\lambda \geq \tilde{\lambda}_0$, then

$$G_{b'}^-(b,b',\lambda) - \frac{\beta}{2}|G_b^-(b',b'',\lambda)| \geq \varepsilon/\lambda^\eta \tag{73}$$

and

$$G_{b'}^+(b,b',\lambda) - \frac{\beta}{2}|G_b^+(b',b'',\lambda)| \geq \varepsilon/\lambda^\eta \tag{74}$$

for every (b,b',b''). Observe that the left-hand side of (71) is bounded away from zero, while the left-hand side of (73) may fall to zero as λ goes to ∞, provided that such a decline does not happen too fast. A similar remark applies to (72) and (74).

Our next step consists of characterizing the part of the expected payoff of the date-t incumbent that depends on that player's actions. Given that each incumbent faces a problem similar to the ones faced by its predecessors and successors, it suffices to carry out that task for party p_0 when the initial public debt assumes a generic value b_0.

Let Ω_{p_0} denote the expected payoff of the date-zero incumbent and ω_{p_0} be the part of Ω_{p_0} that depends on that player's actions. We define $\omega_{p_0,t}$ as the undiscounted date-t part of

ω_{p_0}. Thus, $\omega_{p_0} = \sum_{t=0}^{\infty} \beta^t \omega_{p_0,t}$. To assess ω_{p_0}, we evaluate each of the factors $\omega_{p_0,t}$. At date zero, party p_0 chooses b_1 and its date-zero period payoff is equal to $U(b_0, G(b_0, b_1, \lambda), b_1) + \lambda G(b_0, b_1, \lambda)$. Hence, $\omega_{p_0,0}$ is equal to that expression.

With respect to date 1, if the date-zero incumbent p_0 were again in office, then its period payoff would be $U(b_1, G(b_1, b_2, \lambda), b_2) + \lambda G(b_1, b_2, \lambda)$; otherwise, the party in office would leave the debt at $\bar{B}$ and the period payoff of party p_0 would be $U(b_1, G(b_1, \bar{B}, \lambda), \bar{B})$. Hence,

$$\omega_{p_0,1} = \frac{1}{n} \left[U(b_1, G(b_1, b_2, \lambda), b_2) + \lambda G(b_1, b_2, \lambda) \right] + \frac{n-1}{n} U(b_1, G(b_1, \bar{B}, \lambda), \bar{B}).$$

At date 2, suppose that p_0 were in office at date 1. If it were again in office at $t = 2$, then its payoff would be $U(b_2, G(b_2, b_3, \lambda), b_3) + \lambda G(b_2, b_3, \lambda)$; otherwise, its period payoff would be $U(b_2, G(b_2, \bar{B}, \lambda), \bar{B})$. Hence, the term

$$\frac{1}{n} \left\{ \frac{1}{n} \left[U(b_2, G(b_2, b_3, \lambda), b_3) + \lambda G(b_2, b_3, \lambda) \right] + \frac{n-1}{n} U(b_2, G(b_2, \bar{B}, \lambda), \bar{B}) \right\} \tag{75}$$

must be a component of $\omega_{p_0,2}$. Suppose now that party p_0 were not in office at date 1; its period payoff would be $U(\bar{B}, G(\bar{B}, \bar{B}, \lambda), \bar{B}) + \lambda G(\bar{B}, \bar{B}, \lambda)$ if it were in office at date 2 and $U(\bar{B}, G(\bar{B}, \bar{B}, \lambda), \bar{B})$ otherwise. Since these last expressions do not depend on the choices of party p_0, we conclude that $\omega_{p_0,2}$ is equal to the expression in (75).

We now apply this reasoning to a generic date $t \geq 2$. Suppose that p_0 were in office at all previous dates. If it were again in office, then its period payoff would be equal to $U(b_t, G(b_t, b_{t+1}, \lambda), b_{t+1}) + \lambda G(b_t, b_{t+1}, \lambda)$; otherwise, its period payoff would be equal to $U(b_t, G(b_t, \bar{B}, \lambda), \bar{B})$. If p_0 were not in office on at least one of the previous dates, then its period payoff would be $U(\bar{B}, G(\bar{B}, \bar{B}, \lambda), \bar{B}) + \lambda G(\bar{B}, \bar{B}, \lambda)$ if it were in office at date t and $U(\bar{B}, G(\bar{B}, \bar{B}, \lambda), \bar{B})$ otherwise. Therefore,

$$\begin{aligned} \omega_{p_0,t} \;=\; & \left(\frac{1}{n}\right)^{t-1} \left\{ \frac{1}{n} [U(b_t, G(b_t, b_{t+1}, \lambda), b_{t+1}) + \right. \\ & \left. \lambda G(b_t, b_{t+1}, \lambda)] + \frac{n-1}{n} U(b_t, G(b_t, \bar{B}, \lambda), \bar{B}) \right\}. \end{aligned} \tag{76}$$

We conclude that

$$\begin{aligned} \omega_{p_0} \;=\; & U(b_0, G(b_0, b_1, \lambda), b_1) + \lambda G(b_0, b_1, \lambda) \\ & \beta \left\{ \frac{1}{n} \left[U(b_1, G(b_1, b_2, \lambda), b_2) + \lambda G(b_1, b_2, \lambda) \right] + \right. \\ & \left. \frac{n-1}{n} U(b_1, G(b_1, \bar{B}, \lambda), \bar{B}) \right\} + \sum_{t=2}^{\infty} \beta^t \omega_{p_0,t}. \end{aligned} \tag{77}$$

For future reference, we point out that $\sum_{t=2}^{\infty} \beta^t \omega_{p_0,t}$ does not depend on b_1.

Lemma 8 *Suppose that (73) and (74) hold. Then, there exists a real number $\tilde{\lambda}$ with the property that, if $\lambda \geq \tilde{\lambda}$, then ω_{p_0} is strictly increasing in b_{t+1} for every t.*

Proof. Let $\frac{\partial \omega_{p_0}}{\partial b_1^-}$ and $\frac{\partial \omega_{p_0}}{\partial b_1^+}$ denote the left- and right-side partial derivatives of ω_{p_0} with respect to b_1. Hence,

$$\begin{aligned}\frac{\partial \omega_{p_0}}{\partial b_1^-} &= U_{b'}(b_0, G(b_0, b_1, \lambda), b_1) + U_g(b_0, G(b_0, b_1, \lambda), b_1)G_{b'}^-(b_0, b_1, \lambda) + \\ &\quad \lambda G_{b'}^-(b_0, b_1, \lambda) + \beta\frac{1}{n}[U_b(b_1, G(b_1, b_2, \lambda), b_2) + \\ &\quad U_g(b_1, G(b_1, b_2, \lambda), b_2)G_b^-(b_1, b_2, \lambda) + \lambda G_b^-(b_1, b_2, \lambda)] + \\ &\quad \beta\frac{n-1}{n}[U_b(b_1, G(b_1, \bar{B}, \lambda), \bar{B}) + U_g(b_1, G(b_1, \bar{B}, \lambda), \bar{B})G_b^-(b_1, \bar{B}, \lambda)].\end{aligned} \tag{78}$$

Use the fact that $G_b^- \leq 0$ and $U_{b'} \geq 0$ to conclude that

$$\begin{aligned}\frac{\partial \omega_{p_0}}{\partial b_1^-} &\geq U_g(b_0, G(b_0, b_1, \lambda), b_1)G_{b'}^-(b_0, b_1, \lambda) + \\ &\quad \beta\frac{1}{n}U_b(b_1, G(b_1, b_2, \lambda), b_2) + \beta\frac{n-1}{n}U_b(b_1, G(b_1, \bar{B}, \lambda), \bar{B}) + \\ &\quad \beta\frac{1}{n}U_g(b_1, G(b_1, b_2, \lambda), b_2)G_b^-(b_1, b_2, \lambda) + \\ &\quad \beta\frac{n-1}{n}U_g(b_1, G(b_1, \bar{B}, \lambda), \bar{B})G_b^-(b_1, \bar{B}, \lambda) + \\ &\quad \lambda\left[G_{b'}^-(b_0, b_1, \lambda) - \beta\frac{1}{n}|G_b^-(b_1, b_2, \lambda)|\right].\end{aligned}$$

Now, observe that $U_{gg} \leq 0$, $G_{b'}^- \geq 0$, $U_{bg} \leq 0$. Therefore,

$$\begin{aligned}\frac{\partial \omega_{p_0}}{\partial b_1^-} &\geq U_g(b_0, \bar{\Gamma}, b_1)G_{b'}^-(b_0, b_1, \lambda) + \\ &\quad \beta\left\{\frac{1}{n}U_b(b_1, \bar{\Gamma}, b_2) + \frac{n-1}{n}U_b(b_1, \bar{\Gamma}, \bar{B})\right\} + \\ &\quad \beta\left\{\frac{1}{n}U_g(b_1, \gamma, b_2)G_b^-(b_1, b_2, \lambda) + \frac{n-1}{n}U_g(b_1, \gamma, \bar{B})G_b^-(b_1, \bar{B}, \lambda)\right\} + \\ &\quad \lambda\left[G_{b'}^-(b_0, b_1, \lambda) - \beta\frac{1}{n}|G_b^-(b_1, b_2, \lambda)|\right].\end{aligned}$$

The last expression implies that

$$\begin{gathered}\frac{\partial \omega_{p_0}}{\partial b_1^-} \geq \left[\min_{(b,b')} U_g(b, \bar{\Gamma}, b')\right] G_{b'}^-(b_0, b_1, \lambda) + \beta \min_{(b,b')} U_b(b, \bar{\Gamma}, b') + \\ \beta\left\{\frac{1}{n}\left[\max_{(b,b')} U_g(b, \gamma, b')\right] G_b^-(b_1, b_2, \lambda) + \frac{n-1}{n}\left[\max_{(b,b')} U_g(b, \gamma, b')\right] G_b^-(b_1, \bar{B}, \lambda)\right\} + \\ \lambda\left[G_{b'}^-(b_0, b_1, \lambda) - \beta\frac{1}{n}|G_b^-(b_1, b_2, \lambda)|\right] > -\infty.\end{gathered}$$

Since $n \geq 2$, $U_g(b, \bar{\Gamma}, b') < 0$, $G^-_{b'} \geq 0$, and $U_g(b, \gamma, b') > 0$, we have

$$\frac{\partial \omega_{p_0}}{\partial b_1^-} \geq A^- + \lambda \left[G^-_{b'}(b_0, b_1, \lambda) - \beta \frac{1}{2} |G^-_b(b_1, b_2, \lambda)| \right], \tag{79}$$

where

$$\begin{aligned} A^- \quad = \quad & \left[\min_{(b,b')} U_g(b, \bar{\Gamma}, b') \right] \left[\sup_{(b,b',\lambda)} G^-_{b'}(b, b', \lambda) \right] + \beta \min_{(b,b')} U_b(b, \bar{\Gamma}, b') - \\ & \beta \left[\max_{(b,b')} U_g(b, \gamma, b') \right] \left[\sup_{(b,b',\lambda)} |G^-_b(b, b', \lambda)| \right]. \end{aligned} \tag{80}$$

Now use the fact that b and b' belong to $[-\underline{B}, \bar{B}]$ and that the partial derivatives of U are continuous to conclude that $\min_{(b,b')} U_g(b, \bar{\Gamma}, b') > -\infty$, $\min_{(b,b')} U_b(b, \bar{\Gamma}, b') > -\infty$, and $\max_{(b,b')} U_g(b, \gamma, b') < \infty$. Therefore, (69) and (70) imply that $A^- > -\infty$.

Combine (79) with (73) to conclude that if $\lambda > \tilde{\lambda}_0$, then

$$\frac{\partial \omega_{p_0}}{\partial b_1^-} \geq A^- + \lambda^{1-\eta} \varepsilon > -\infty.$$

Similar reasoning establishes

$$\frac{\partial \omega_{p_0}}{\partial b_1^+} \geq A^+ + \lambda^{1-\eta} \varepsilon > -\infty,$$

where A^+ is defined exactly as A^-, except that $G^+_{b'}$ and G^+_b replace their left-sided counterparts in (80). Thus, there exists a number $\tilde{\lambda} \geq \tilde{\lambda}_0$ with the property that if $\lambda > \tilde{\lambda}$, then $\frac{\partial \omega_{p_0}}{\partial b_1^-} > 0$ and $\frac{\partial \omega_{p_0}}{\partial b_1^+} > 0$. Since ω_{p_0} is a continuous function of b_1, we conclude that ω_{p_0} is strictly increasing in b_1.

We still have to show that ω_{p_0} is strictly increasing in b_{t+1} for a generic date t. From (77) we conclude that

$$\frac{\partial \omega_{p_0}}{\partial b_{t+1}^-} = \beta^t \frac{\partial \omega_{p_{0,t}}}{\partial b_{t+1}^-} + \beta^{t+1} \frac{\partial \omega_{p_{0,t+1}}}{\partial b_{t+1}^-}.$$

Combine this expression with (76) to obtain

$$\begin{aligned} \frac{\partial \omega_{p_0}}{\partial b_{t+1}^-} = & \left(\frac{\beta}{n} \right)^t [U_{b'}(b_t, G(b_t, b_{t+1}, \lambda), b_{t+1}) + U_g(b_t, G(b_t, b_{t+1}, \lambda), b_{t+1}) G^-_{b'}(b_t, b_{t+1}, \lambda) + \\ & \lambda G^-_{b'}(b_t, b_{t+1}, \lambda)] + \left(\frac{\beta}{n} \right)^t \Big\{ \beta \frac{1}{n} [U_b(b_{t+1}, G(b_{t+1}, b_{t+2}, \lambda), b_{t+2}) + \\ & U_g(b_{t+1}, G(b_{t+1}, b_{t+2}, \lambda), b_{t+2}) G^-_b(b_{t+1}, b_{t+2}, \lambda) + \lambda G^-_b(b_{t+1}, b_{t+2}, \lambda)] + \\ & \beta \frac{n-1}{n} [U_b(b_{t+1}, G(b_{t+1}, \bar{B}, \lambda), \bar{B}) + U_g(b_{t+1}, G(b_{t+1}, \bar{B}, \lambda), \bar{B}) G^-_b(b_{t+1}, \bar{B}, \lambda)] \Big\}. \end{aligned}$$

If we follow the reasoning used after obtaining equality (78), we conclude that

$$\frac{\partial \omega_{p_0}}{\partial b_{t+1}^-} \geq \left(\frac{\beta}{n}\right)^t (A^- + \lambda^{1-\eta}\varepsilon) > -\infty$$

and

$$\frac{\partial \omega_{p_0}}{\partial b_{t+1}^+} \geq \left(\frac{\beta}{n}\right)^t (A^+ + \lambda^{1-\eta}\varepsilon) > -\infty.$$

Hence, $\frac{\partial \omega_{p_0}}{\partial b_{t+1}^-} > 0$ and $\frac{\partial \omega_{p_0}}{\partial b_{t+1}^+} > 0$ for $\lambda > \tilde{\lambda}$. An appeal to continuity establishes that ω_{p_0} is strictly increasing in b_{t+1}. ■

We can now finally establish the main result of this section.

Proposition 8 *Suppose that (73) and (74) hold. If $\lambda \geq \tilde{\lambda}$, then the spendthrift policy plan $\{\tilde{\sigma}_t\}_{t=0}^{\infty}$ is a symmetric political equilibrium.*

Proof. Let t be any date. We have to show that if party p_t believes that the other parties follow the strategy $\{\tilde{\sigma}_s\}_{s=0}^{\infty}$, then $\{\tilde{\sigma}_s\}_{s=0}^{\infty}$ is an optimal choice for p_t. It is enough to consider the situation of party p_0 when the initial public debt has a generic value b_0. The problem of party p_0 consists of selecting a sequence $\{b_{t+1}\}_{t=0}^{\infty}$ that maximizes ω_{p_0} subject to

$$b_{t+1} \leq \bar{B}, \tag{81}$$
$$f^b(b_t) \leq b_{t+1}. \tag{82}$$

Let $\{\hat{b}_{t+1}\}_{t=0}^{\infty}$ be any sequence that satisfies (81) and (82) with the property that $\hat{b}_1 < \bar{B}$. We will show that such a sequence cannot solve the problem of party p_0 by constructing a sequence $\{b_{t+1}\}_{t=0}^{\infty}$ that satisfies these constraints and yields a higher payoff.

Let b_1 be any debt level that satisfies $\hat{b}_1 < b_1 \leq \bar{B}$. Define the debt level at the other dates recursively according to

$$b_{t+1} = \max\{f^b(b_t), \hat{b}_{t+1}\} \tag{83}$$

Therefore, $\{b_{t+1}\}_{t=0}^{\infty}$ satisfies (82).

We next show that $\{b_{t+1}\}_{t=0}^{\infty}$ satisfies (81). Recall that f^b is strictly increasing and $f^b(\bar{B}) = \bar{B}$. Thus, the inequality $b_1 \leq \bar{B}$ implies that $f^b(b_1) \leq f^b(\bar{B}) = \bar{B}$. Since $\hat{b}_2 \leq \bar{B}$, we conclude that $\max\{f^b(b_1), \hat{b}_2\} \leq \bar{B}$. Thus, $b_2 \leq \bar{B}$. Apply this reasoning recursively to conclude that $\{b_{t+1}\}_{t=0}^{\infty}$ satisfies (81).

To conclude the proof, observe that (83) implies that $b_{t+1} \geq \hat{b}_{t+1}$. Therefore, an appeal to Lemma 8 establishes that $\{b_{t+1}\}_{t=0}^{\infty}$ yields a higher payoff than $\{\hat{b}_{t+1}\}_{t=0}^{\infty}$. Hence, the optimal action for party p_0 entails setting b_1 equals to $\bar{B}$. Therefore, $\{\tilde{\sigma}_t\}_{t=0}^{\infty}$ is an optimal strategy for the date-zero incumbent. ■

II.3 The assumptions about the derivatives of G

In this subsection we show that the assumptions on the partial derivatives of the function G introduced in (66), (71), (72), (73) and (74) are satisfied in the context of example 2.

We start by pointing out that the analysis can be restricted to (71) and (72). Indeed, if the first of these two conditions holds, then so does (73). To verify that, it is enough to set $\eta = 0$ in (73). Similarly, (74) holds whenever (72) is satisfied.

Concerning (66), recall that it was used only to argue in an intuitive way that the spendthrift policy is an equilibrium outcome. As discussed in the previous subsection of this online appendix, a formal analysis of the spendthrift equilibrium requires that we replace the two partial derivatives $G_{b'}$ and G_b in inequality (66) by their left and right counterparts: $-G^-_{b'}(b, b', \lambda)/G^-_b(b', b'', \lambda) > \beta/2$ and $-G^+_{b'}(b, b', \lambda)/G^+_b(b', b'', \lambda) > \beta/2$. Suppose that (71) holds. Thus,

$$-\frac{G^-_{b'}(b, b', \lambda)}{G^-_b(b', b'', \lambda)} = \frac{G^-_{b'}(b, b', \lambda)}{|G^-_b(b', b'', \lambda)|} \geq \frac{\varepsilon}{|G^-_b(b', b'', \lambda)|} + \frac{\beta}{2} > \frac{\beta}{2}.$$

A similar argument establishes that $-G^+_{b'}(b, b', \lambda)/G^+_b(b', b'', \lambda) > \beta/2$ if (72) holds.

Our task from now on is to show that both (71) and (72) are satisfied. To achieve this goal, it suffices to introduce just one of the following two assumptions:

$$\lambda \geq \frac{a_1 + a_2}{1 - \bar{\tau}} \tag{84}$$

or

$$a_1 < a_3. \tag{85}$$

Next we show that each of them implies that both (71) and (72) hold and, as a consequence, so do the left and right counterparts of (66).

The sufficiency of inequality (84). If (84) holds, then $G(b, b', \lambda) = \Gamma(b, b')$ for all (b, b'). That is, λ is sufficiently large so that the maximization of $V(b, g, b')$ will always lead to a corner solution. To verify that, observe that

$$V_g(b, \Gamma(b, b'), b') = U_g(b, \Gamma(b, b'), b') + \lambda = -\frac{a_1 + a_2}{1 - \bar{\tau}} + \frac{a_3}{\Gamma(b, b')} + \lambda.$$

Now use the fact that $a_3/\Gamma(b, b') > 0$ to conclude that

$$V_g(b, \Gamma(b, b'), b') > -\frac{a_1 + a_2}{1 - \bar{\tau}} + \lambda.$$

Thus, (84) implies that $V_g(b, \Gamma(b, b'), b') > 0$. Given that $V_{gg} < 0$, $G(b, b', \lambda) = \Gamma(b, b')$.

Since Γ is differentiable, in such a context both (71) and (72) are equivalent to

$$\Gamma_{b'}(b, b') - \frac{\beta}{2}|\Gamma_b(b', b'')| \geq \varepsilon. \tag{86}$$

Therefore, our task consists of finding such a positive ε. However, $\Gamma_{b'} = \beta$ and $\Gamma_b = -1$. Thus, the left-hand side of (86) is equal to $\beta/2$. Hence, it is enough to set $\varepsilon = \beta/2$.

The sufficiency of inequality (85). Since we are not assuming that (84) holds, it is not possible to be sure that $G(b, b', \lambda)$ is a corner solution. Thus, we must also take into consideration the case in which V has an interior optimum.

Recall that G^u denotes the unconstrained maximizer of V. Therefore, $G(b, b', \lambda)$ is equal to $\Gamma(b, b')$ or $G^u(b, b', \lambda)$. The function G may fail to be differentiable exactly when $\Gamma(b, b') = G^u(b, b', \lambda)$; however, these two last functions are differentiable. Hence, G_b^- must be equal to Γ_b or G_b^u; the same is true for G_b^+. Similarly, both G_b^+ and $G_{b'}^+$ must be equal to $\Gamma_{b'}$ or $G_{b'}^u$. Thus, to show that both (71) and (72) hold, it suffices to find a positive ε with the property that (86),

$$\Gamma_{b'}(b, b') - \frac{\beta}{2}|G_b^u(b', b'', \lambda)| \geq \varepsilon, \tag{87}$$

$$G_{b'}^u(b, b', \lambda) - \frac{\beta}{2}|\Gamma_b(b', b'')| \geq \varepsilon, \tag{88}$$

and

$$G_{b'}^u(b, b', \lambda) - \frac{\beta}{2}|G_b^u(b', b'', \lambda)| \geq \varepsilon \tag{89}$$

are satisfied for all attainable (b, b', b'') and all λ. We deal with each of those four inequalities separately.

We first consider inequality (86). Define $\varepsilon_1 \equiv \beta/2$. Given the analysis carried out in the previous subsection, it is clear that $\Gamma_{b'} - \frac{\beta}{2}|\Gamma_b| \geq \varepsilon_1$. Hence, (86) is satisfied for any positive $\varepsilon \leq \varepsilon_1$.

To study inequalities (87), (88) and (89), we need to evaluate the partial derivatives of G^u. We do so by using the implicit derivatives in (67), as well as some of the derivatives in (56)-(61). For future reference, observe that

$$0 < \frac{U_{bg}(b, g, b')}{U_{gg}(b, g, b')} = \frac{1}{1 - \frac{1}{U_{bg}(b,g,b')} \frac{a_3}{g^2}} < 1, \tag{90}$$

where the equality is obtained by combining (59) and (60) and the inequalities follow from the fact that $U_{gg} < 0$ and $U_{bg} < 0$.

We now turn our attention to inequality (87). Since $\Gamma_{b'} = \beta$, we can use (67) and (90) to conclude that

$$\Gamma_{b'}(b, b') - \frac{\beta}{2}|G_b^u(b', b'', \lambda)| = \beta - \frac{\beta}{2} \frac{1}{1 - \frac{1}{U_{bg}(b,g,b')} \frac{a_3}{g^2}} > \beta - \frac{\beta}{2} = \frac{\beta}{2}.$$

Therefore, it suffices to take any positive $\varepsilon \leq \varepsilon_1$.

To analyze the remaining two inequalities, we will use the fact that

$$G_{b'}^u(b, b', \lambda) > \beta \frac{a_3}{a_1 + a_3}. \tag{91}$$

To show that the last inequality is true, we first show that the ratio U_{bg}/U_{gg} is strictly increasing in g. Define $m_2(m_1) \equiv (1 + m_1)^{-1}$ and

$$m_1(b, g, b') \equiv -\frac{1}{U_{bg}(b, g, b')} \frac{a_3}{g^2}.$$

Thus,

$$\frac{\partial m_1}{\partial g} = a_3 \frac{2U_{bg}g + \frac{\partial U_{bg}}{\partial g} g^2}{(U_{bg}g^2)^2}. \tag{92}$$

Together, (56) and (60) lead to

$$U_{bg} = -\frac{1}{a_1}(U_b)^2 \Rightarrow \frac{\partial U_{bg}}{\partial g} = -\frac{2}{a_1} U_b U_{bg}.$$

Combine the last equality with (92) to conclude that

$$\frac{\partial m_1}{\partial g} = a_3 \frac{2U_{bg}g - \frac{2}{a_1} U_b U_{bg} g^2}{(U_{bg}g^2)^2}.$$

Since $U_b < 0$ and $U_{bg} < 0$, we conclude that $\frac{\partial m_1}{\partial g} < 0$. Finally, we use the facts that $\frac{dm_2}{dm_1} < 0$ and $\frac{\partial (U_{bg}/U_{gg})}{\partial g} = \frac{dm_2}{dm_1} \frac{\partial m_1}{\partial g}$ to establish that $\frac{\partial (U_{bg}/U_{gg})}{\partial g} > 0$.

Now recall that $g^u(b, b')$ is the unconstrained maximizer of $U(b, g, b')$. Since $g^u(b, b') < G^u(b, b', \lambda)$,

$$\beta \frac{U_{bg}(b, G^u(b, b', \lambda), b')}{U_{gg}(b, G^u(b, b', \lambda), b')} > \beta \frac{U_{bg}(b, g^u(b, b'), b')}{U_{gg}(b, g^u(b, b'), b')}.$$

From (60) and (61), $-U_{gb'} = \beta U_{bg}$. Hence,

$$-\frac{U_{gb'}(b, G^u(b, b', \lambda), b')}{U_{gg}(b, G^u(b, b', \lambda), b')} > -\frac{U_{gb'}(b, g^u(b, b'), b')}{U_{gg}(b, g^u(b, b'), b')}.$$

Combine this result with (67) to conclude that

$$G^u_{b'}(b, b', \lambda) = -\frac{U_{gb'}(b, G^u(b, b', \lambda), b')}{U_{gg}(b, G^u(b, b', \lambda), b')} > -\frac{U_{gb'}(b, g^u(b, b'), b')}{U_{gg}(b, g^u(b, b'), b')}. \tag{93}$$

The differentiation of the first-order condition $U_g(b, g^u, b') = 0$ shows that $g^u_{b'} = -U_{gb'}/U_{gg}$. Moreover, we know from (64) that $g^u_{b'} = \beta a_3/(a_1 + a_3)$. Combine the last two equalities with (93). This yields (91).

We are now able to show that (88) holds. Combine (91) and the equality $\Gamma_b = -1$. Thus,

$$G^u_{b'}(b, b', \lambda) - \frac{\beta}{2}|\Gamma_b(b', b'')| > \beta \frac{a_3}{a_1 + a_3} - \frac{\beta}{2} = \beta \left(\frac{a_3}{a_1 + a_3} - \frac{1}{2} \right).$$

Define $\varepsilon_2 \equiv \beta \left(\frac{a_3}{a_1 + a_3} - \frac{1}{2} \right)$. Since (85) ensures that $\varepsilon_2 > 0$, any positive $\varepsilon \leq \varepsilon_2$ satisfies (88).

Finally, we study inequality (89). We adopt a similar approach. Observe that

$$G^u_{b'}(b, b', \lambda) - \frac{\beta}{2}|G^u_b(b', b'', \lambda)| > \beta \frac{a_3}{a_1 + a_3} - \frac{\beta}{2} \frac{U_{bg}(b, G^u(b, b', \lambda), b')}{U_{gg}(b, G^u(b, b', \lambda), b')}.$$

Recall that (90) establishes that $0 < U_{bg}/U_{gg} < 1$. Therefore,

$$G^u_{b'}(b, b', \lambda) - \frac{\beta}{2}|G^u_b(b', b'', \lambda)| > \beta \frac{a_3}{a_1 + a_3} - \frac{\beta}{2}.$$

As before, (89) is satisfied for any positive $\varepsilon \leq \varepsilon_2$.

To conclude this subsection, we define $\varepsilon \equiv \min\{\varepsilon_1, \varepsilon_2\}$. Clearly, this definition entails that (86), (87), (88) and (89) hold.

III The policy g^N

Consider the problem of maximizing $\Omega(g, 0)$ subject to $g \leq \Gamma(0, 0)$, where $\Omega(g, b)$ is defined in (43). The first-order condition is $\Omega_g(g, 0) \geq 0$. If the constraint does not bind, then that condition becomes equality (49). Thus, it should be clear that the variable g^N defined in subsection 5.2 is the maximizer of $\Omega(g, 0)$ under the assumption that the constraint $g \leq \Gamma(0, 0)$ does not bind.

In Proposition 9 we establish the properties of g^N that relate to the reasoning developed in subsection 5.2. Before that, we state the following lemma.

Lemma 9 *Let $(\beta, U, \gamma, \Gamma, f^b, \bar{B})$ be an economy with $B_L = \bar{B}$. There are numbers λ_1^b and $N^N(\beta, \lambda)$ with the property that, for every polity (λ, n) satisfying $\lambda > \lambda_1^b$, the policy $(g^N, 0)$ is a symmetric political outcome if and only if $n \leq N^N(\beta, \lambda)$. Furthermore, $NGC(g^N) > 0$ if $n < N^N(\beta, \lambda)$.*

Since the proof of this lemma is long, we present it at the end of this section.

Proposition 9 *Suppose that $g^N < \Gamma(0, 0)$. Then:*
(i) When $NGC(g^N) < 0$, $NGC(g) < 0$ for every g.
(ii) If B_L is sufficiently close to zero, $NGC(g^N) \geq 0$.
(iii) If B_L is sufficiently close to $\bar{B}$, $NGC(g^N) \geq 0$ if λ is sufficiently large and n is sufficiently small.

Proof. Since g^N is the maximizer of NGC, $NGC(g^N) \geq NGC(g)$ for every g. Hence, (i) must hold. Concerning (ii), observe that

$$NGC(g; B_L) = \frac{\beta}{1-\beta}\left\{U(0, g, 0) - U(B_L, G(B_L, B_L, \lambda), B_L) + \frac{\lambda}{n}[g - G(B_L, B_L, \lambda)]\right\} - [V(0, G(0, B_L, \lambda), 0) - V(0, g, 0)],$$

where the notation $NGC(g; B_L)$ is used to emphasize that NGC depends on B_L. Set $B_L = 0$. Simple algebraic manipulations then lead to

$$NGC(g; 0) = \Omega(g, 0) - \Omega(G(0, 0, \lambda), 0).$$

Because g^N is an interior maximizer of $\Omega(g, 0)$, which is strictly concave in g, it follows that

$$\Omega(g^N, 0) - \Omega(G(0, 0, \lambda), 0) > 0 \Rightarrow NGC(g^N; 0) > 0.$$

Given the continuity of all relevant functions, $NGC(g^N; B_L) \geq 0$ for $B_L > 0$ but sufficiently close to 0.

Result (iii) would follow from Lemma 9 if $B_L = \bar{B}$. The continuity of NGC then implies that $NGC(g^N; B_L) \geq 0$ whenever B_L is sufficiently close to $\bar{B}$ provided that λ is sufficiently large and n is sufficiently small. ■

More generally, whether $NGC(g^N; B_L) \geq 0$ holds or not for intermediate levels of B_L depends on the array $(\beta, U, \gamma, \Gamma, f^b, \bar{B}, \lambda, n)$ that characterizes a society.

Proof of Lemma 9. Recall that g^N depends on n. We emphasize this fact along this proof by writing $g^N(n)$. Furthermore, functions Ω_0 and Ω also depend on n. Thus, we will also explicitly write n as an argument of these two functions.

From Lemma 6 (see subsection II.2 of this online appendix), $G(0, \bar{B}, \lambda) = \Gamma(0, \bar{B})$ for a sufficiently large λ. Furthermore, $\Gamma(0, 0) \geq g^N(2)$. Therefore,

$$G(0, \bar{B}, \lambda) - g^N(2) \geq \Gamma(0, \bar{B}) - \Gamma(0, 0) > 0$$

for a large λ. Hence, there exists a number λ_1^b (that does not depend on n) such that if $\lambda > \lambda_1^b$, then

$$U(0, \Gamma(0, \bar{B}), \bar{B}) - U(0, g^*(0, 0), 0) + \lambda[G(0, \bar{B}, \lambda) - g^N(2)] > \frac{\beta}{1-\beta}\left[U(0, g^*(0, 0), 0) - U(\bar{B}, \gamma, \bar{B})\right].$$

We use the facts that

$$U(0, \Gamma(0, \bar{B}), \bar{B}) - U(0, g^*(0, 0), 0) + \lambda[G(0, \bar{B}, \lambda) - g^N(2)] = [U(0, \Gamma(0, \bar{B}), \bar{B}) + \lambda G(0, \bar{B}, \lambda)] - [U(0, g^*(0, 0), 0) + \lambda g^N(2)]$$

and $G(0, \bar{B}, \lambda) = \Gamma(0, \bar{B})$ to conclude that

$$[U(0, G(0, \bar{B}, \lambda), \bar{B}) + \lambda G(0, \bar{B}, \lambda)] - [U(0, g^*(0, 0), 0) + \lambda g^N(2)] > \frac{\beta}{1-\beta}\left[U(0, g^*(0, 0), 0) - U(\bar{B}, \gamma, \bar{B})\right] \geq \frac{\beta}{1-\beta}\left[U(0, g^N(n), 0) - U(\bar{B}, \gamma, \bar{B})\right]$$

for every n. These inequalities imply that there is a number $k(\beta, \lambda)$ with the property that if $n \geq k(\beta, \lambda)$, then

$$[U(0, G(0, \bar{B}, \lambda), \bar{B}) + \lambda G(0, \bar{B}, \lambda)] - [U(0, g^*(0, 0), 0) + \lambda g^N(2)] > \frac{\beta}{1-\beta}\left[U(0, g^N(n), 0) - U(\bar{B}, \gamma, \bar{B})\right] + \frac{1}{n}\frac{\beta}{1-\beta}\lambda[\Gamma(0, 0) - \gamma].$$

Combine the last inequality with $U(0, g^*(0, 0), 0) \geq U(0, g^N(n), 0)$ and $\Gamma(0, 0) \geq g^N(2) \geq$

$g^N(n)$ to conclude that

$$[U(0, G(0, \bar{B}, \lambda), \bar{B}) + \lambda G(0, \bar{B}, \lambda)] - [U(0, g^N(n), 0) + \lambda g^N(n)] > \frac{\beta}{1-\beta}\left[U(0, g^N(n), 0) - U(\bar{B}, \gamma, \bar{B})\right] + \frac{1}{n}\frac{\beta}{1-\beta}\lambda[g^N(n) - \gamma].$$

This inequality is equivalent to $\Omega_0(\{\tilde{g}_t, \tilde{b}_{t+1}\}_{t=0}^{\infty}; n) > \Omega(g^N(n), 0; n)$. Thus, if $\lambda > \lambda_1^b$ and $n \geq k(\beta, \lambda)$, then $(g^N(0, n), 0)$ is not a symmetric political outcome.

Consider inequality (42). We use it to conclude that the policy $(g^N(n), 0)$ is an equilibrium outcome if and only if

$$NGC(g^N; n) = \Omega(g^N(n), 0; n) - \Omega_0(\{\tilde{g}_t, \tilde{b}_{t+1}\}_{t=0}^{\infty}; n) \geq 0,$$

where n as argument of NGC emphasizes that it also depends on n. Now, observe that

$$\Omega(g^N(N^b(\beta, \lambda)), 0; N^b(\beta, \lambda)) > \Omega(g^*(0, 0), 0; N^b(\beta, \lambda))$$

and

$$\Omega(g^*(0, 0), 0; N^b(\beta, \lambda)) = \Omega_0(\{\tilde{g}_t, \tilde{b}_{t+1}\}_{t=0}^{\infty}; N^b(\beta, \lambda)).$$

Thus, $NGC(g^N; N^b(\beta, \lambda)) > 0$. On the other hand, $NGC(g^N; k(\beta, \lambda)) < 0$. Hence, the intermediate value theorem implies that there exists a number $N^N(\beta, \lambda)$ satisfying $NGC(g^N; N^N(\beta, \lambda)) = 0$.

We still have to show that $(g^N(n), 0)$ is a symmetric political outcome if and only if $n \leq N^N(\beta, \lambda)$. Regardless of whether the constraint $g^N(n) \leq \Gamma(0, 0)$ binds or not, it is possible to show that

$$\frac{\partial NGC(g^N; n)}{\partial n} = -\frac{\beta\lambda}{(1-\beta)n^2}[g^N(n) - \gamma] < 0.$$

Hence, $NGC(g^N; n) \geq 0$ if and only if $n \leq N^N(\beta, \lambda)$. The same argument establishes that $NGC(g^N; n) > 0$ if $n < N^N(\beta, \lambda)$. ■

IV An equilibrium set

In this section we characterize a set of static equilibrium outcomes and investigate how it depends on n and B_L. As discussed in section 5, that dependence is shaped by the interaction between those two variables.

Recall that inequality (42) defines a necessary and sufficient condition for a policy $\{g_t, b_{t+1}\}_{t=0}^{\infty}$ to be an equilibrium outcome when there is no legal constraint on the government debt. When there is such a constraint, a sufficient condition for a policy $\{g_t, b_{t+1}\}_{t=0}^{\infty}$ to be an equilibrium is

$$\Omega_s(\{g_t, b_{t+1}\}_{t=s}^{\infty}) \geq U(0, G(0, B_L, \lambda), B_L) + \lambda G(0, B_L, \lambda) + \frac{\beta}{1-\beta}\left[U(B_L, G(B_L, B_L, \lambda), B_L) + \frac{\lambda}{n}G(B_L, B_L, \lambda)\right].$$

This inequality is still too open-ended to carry out a sharp analysis of how the interaction of n and B_L impacts the sustainability of a given policy. Hence, as in the main text, we focus on static outcomes here.

To save on notation, denote the right-hand side of the last inequality by $R(n, B_L)$. When its left-hand side is restricted to sequences in which $g_t = g$ and $b_{t+1} = 0$ for all t, it becomes

$$\Omega(g, 0) \geq R(n, B_L), \tag{94}$$

where $\Omega(g, 0)$ is defined in (43).

Let $\mathbb{S}(n, B_L)$ be the set of all values of g that satisfy (94). Recall that g^N is the maximizer of $\Omega(g, 0)$. Hence, no static policy will be sustainable if $\Omega(g^N, 0) < R(n, B_L)$; as a consequence, $\mathbb{S}(n, B_L) = \varnothing$ in that case. If $\Omega(g^N, 0) = R(n, B_L)$, then $\mathbb{S}(n, B_L) = \{g^N\}$. Finally, if $\Omega(g^N, 0) > R(n, B_L)$, then $\mathbb{S}(n, B_L) = [g^s, g^N]$, where g^s solves the equation $\Omega(g^s, 0) = R(n, B_L)$. Observe that if $g^s > g^*(0, 0)$, then $g^s = g^f$, which was defined in subsection 5.2.

We now start to evaluate how changes in n impact $\mathbb{S}(n, B_L)$. For simplicity, we assume throughout the remainder of this section that g^N is an interior maximizer of $\Omega(g, 0)$. For future reference, observe that

$$NGC(g, n, B_L) = \Omega(g, 0) - R(n, B_L).$$

$NGC(g, n, B_L)$ is exactly the net gain from a political compromise defined in section 5. We add n and B_L to the list of arguments to emphasize that NGC also depends on those variables.

Consider first the case in which

$$NGC(g^N, n, B_L) < 0. \tag{95}$$

That is equivalent to saying that $\mathbb{S}(n, B_L) = \varnothing$. Let NGC_n denote the partial derivative of NGC with respect to n. We have that

$$NGC_n(g^N, n, B_L) = \frac{\partial \Omega(g^N, 0)}{\partial g} \frac{\partial g^N}{\partial n} + \frac{\beta \lambda}{(1 - \beta) n^2} [G(B_L, B_L, \lambda) - g^N].$$

However, g^N is an interior maximizer of Ω. Therefore, $\partial \Omega(g^N, 0)/\partial g = 0$ and

$$NGC_n(g^N, n, B_L) = \frac{\beta \lambda}{(1 - \beta) n^2} [G(B_L, B_L, \lambda) - g^N]. \tag{96}$$

Thus, $NGC_n(g^N, n, B_L)$ will have the same sign as the difference $G(B_L, B_L, \lambda) - g^N$. Hence, if $G(B_L, B_L, \lambda) - g^N \leq 0$, then (95) will continue holding after the increase in n, and $\mathbb{S}(n, B_L)$ will continue to be empty after such an increase. On the other hand, that set may become not empty if n decreases. The opposite happens when $G(B_L, B_L, \lambda) - g^N > 0$.

We now turn to the case in which

$$NGC(g^N, n, B_L) = 0, \tag{97}$$

so that $\mathbb{S}(n, B_L) = \{g^N\}$. We again resort to (96). If $G(B_L, B_L, \lambda) - g^N > 0$, then an increase in n will make the left-hand side of (97) larger than its right-hand side. Therefore, such an increase will expand the set $\mathbb{S}(n, B_L)$. On the other hand, the set will become empty in n decreases. The opposite happens when $G(B_L, B_L, \lambda) - g^N \leq 0$.

Finally, consider now the case in which $NGC(g^N, n, B_L) > 0$. As pointed out above, in this case $\mathbb{S}(n, B_L) = [g^s, g^N]$. Hence, we need to assess the impact of n over both endpoints of that interval. Consider first g^N. Since $\partial g^N / \partial n < 0$, an increase in n will decrease the value of that endpoint. Concerning g^s, recall that

$$\Omega(g^s, 0) = R(n, B_L) \Leftrightarrow NGC(g^s, n, B_L) = 0.$$

Differentiating the last equality with respect to n, we obtain

$$\frac{\partial NGC}{\partial g} \frac{\partial g^s}{\partial n} + \frac{\partial NGC}{\partial n} = 0 \Rightarrow \frac{\partial g^s}{\partial n} = -\frac{\frac{\partial NGC}{\partial n}}{\frac{\partial NGC}{\partial g}}.$$

Hence,

$$\frac{\partial g^s}{\partial n} = \frac{\beta \lambda}{(1-\beta) n^2} \frac{g^s - G(B_L, B_L, \lambda)}{\Omega_g(g^s, 0)}.$$

Use the facts that Ω is strictly concave in g, $g^s < g^N$ and $\Omega_g(g^N, 0) = 0$ to conclude that $\Omega_g(g^s, 0) > 0$. Thus, $\partial g^s / \partial n$ will have the same sign as the difference $g^s - G(B_L, B_L, \lambda)$. Furthermore, $\partial g^s / \partial n$ is strictly increasing in B_L.

The main message from this exercise is that the impact of a change in n on the set $\mathbb{S}(n, B_L)$ depends on the value of $G(B_L, B_L, \lambda)$, which in turn hinges on B_L. The effects are entirely analogous to those we analyze in section 5 of the main text, underlining that the set in question depends on the interaction between n and B_L in a very precise and yet general way.