Online Supplement to the Paper

Equilibrium Multiplicity and Uniqueness Under Inflation Targeting

Alexandre B. Cunha

Federal University of Rio de Janeiro

This document is divided into two sections, Appendices A and B. Equation numbers continue sequentially from the last number used in the main body of the paper.

Appendix A: propositions, proofs, and further technical content

Lemma 1 *The following five statements are true:*
(i) $\max_{\pi\in\Pi} U(\pi^e,\pi) = \max\{U(\pi^e,\pi^*), U(\pi^e, f(\pi^e))\}$*;*
(ii) $\max\{U(\pi^e,\pi^*), U(\pi^e, f(\pi^e))\} = \max\{W(\pi^e,\pi^*), W(\pi^e, f(\pi^e)) - C\}$*;*
(iii) if $U(\pi^e,\pi^*) > U(\pi^e, f(\pi^e))$*, then* π^* *is the unique solution of (9);*
(iv) if $U(\pi^e,\pi^*) < U(\pi^e, f(\pi^e))$*, then* $f(\pi^e)$ *is the unique solution of (9);*
(v) if $U(\pi^e,\pi^*) = U(\pi^e, f(\pi^e))$*, then* π^* *and* $f(\pi^e)$ *are the only two solutions of (9).*

Proof. I start with statement (i). Let π^e be any element of Π. Then,

$$\pi \neq \pi^* \implies U(\pi^e,\pi) = W(\pi^e,\pi) - C \leq W(\pi^e, f(\pi^e)) - C \leq U(\pi^e, f(\pi^e)).$$

Hence,

$$\pi \neq \pi^* \implies U(\pi^e,\pi) \leq \max\{U(\pi^e,\pi^*), U(\pi^e, f(\pi^e))\}. \tag{21}$$

Since $U(\pi^e,\pi^*) \leq \max\{U(\pi^e,\pi^*), U(\pi^e, f(\pi^e))\}$, the inequality in (21) holds for all $\pi \in \Pi$; therefore, (i) is true. Given that $U(\pi^e,\pi^*) = W(\pi^e,\pi^*)$ and

$$f(\pi^e) \neq \pi^* \implies U(\pi^e, f(\pi^e)) = W(\pi^e, f(\pi^e)) - C,$$

the equality in (ii) holds if $f(\pi^e) \neq \pi^*$. If $f(\pi^e) = \pi^*$, then both sides of the equality in question will be equal to $W(\pi^e,\pi^*)$. Hence, statement (ii) is true. The last three statements follow directly from (i). ∎

Remark 2 Observe that k_1 is exactly the value of C for which $U(\pi^*, f(\pi^*)) = U(\pi^*, \pi^*)$. Hence, if $C = k_1$ and player p (the general public) selects $\pi^e = \pi^*$, then player b (the central bank) will be indifferent between the actions $f(\pi^*)$ and π^*. Similarly, $U(\hat{\pi}, f(\hat{\pi})) = U(\hat{\pi}, \pi^*)$ for $C = k_2$. Hence, if the last equality holds, then b can optimally play either $f(\hat{\pi})$ or π^* as a response to $\pi^e = \hat{\pi}$.

Lemma 3 *The parameters k_1 and k_2 satisfy the inequality $k_1 < k_2$.*

Proof. Define the function $\psi(\pi^e)$ according to

$$\psi(\pi^e) = W(\pi^e, f(\pi^e)) - W(\pi^e, \pi^*). \tag{22}$$

Thus, $\psi'(\pi^e) = W_1(\pi^e, f(\pi^e)) + W_2(\pi^e, f(\pi^e))f'(\pi^e) - W_1(\pi^e, \pi^*)$. Together, the last equality and (5) imply that $\psi'(\pi^e) = W_1(\pi^e, f(\pi^e)) - W_1(\pi^e, \pi^*)$. Since $W_1(\pi^e, \pi) = 2\alpha[\alpha(\pi - \pi^e) - \mu\bar{y}]$, $\psi'(\pi^e) = 2\alpha^2[f(\pi^e) - \pi^*]$. On the other hand, (6) and (7) imply that

$$f(\pi^*) - \pi^* = \frac{\gamma}{\gamma + \alpha^2}(\hat{\pi} - \pi^*) > 0.$$

Given that f is strictly increasing, $f(\pi^e) - \pi^* \geq f(\pi^*) - \pi^* > 0$ for every $\pi^e \geq \pi^*$. Thus,

$$\psi'(\pi^e) > 0, \forall \pi^e \geq \pi^*. \tag{23}$$

Therefore, $\psi(\pi^*) < \psi(\hat{\pi})$. Since $\psi(\pi^*) = k_1$ and $\psi(\hat{\pi}) = k_2$, it follows that $k_1 < k_2$. ∎

Proposition 4 *If $C < k_1$, then $(\hat{\pi}, \hat{\pi})$ is the unique stage Nash equilibrium. If $C > k_2$, then (π^*, π^*) is the unique stage Nash equilibrium. If $C \in [k_1, k_2]$, then $(\hat{\pi}, \hat{\pi})$ and (π^*, π^*) are the only stage Nash equilibria.*

Proof. Suppose that $C < k_1$. Combine this inequality with Lemma 3 to conclude that $C < k_2$. Then use (11) to show that $W(\hat{\pi}, \pi^*) < W(\hat{\pi}, f(\hat{\pi})) - C$, which in turn implies that $U(\hat{\pi}, \pi^*) < U(\hat{\pi}, f(\hat{\pi}))$. Thus, if p plays $\pi^e = \hat{\pi}$, then the best action for the central bank consists in playing $\pi = f(\hat{\pi})$. Since $f(\hat{\pi}) = \hat{\pi}$, $(\hat{\pi}, \hat{\pi})$ is a stage Nash equilibrium. Concerning uniqueness, optimality for player p implies that $(\hat{\pi}, \pi)$ is not an equilibrium for any $\pi \neq \hat{\pi}$. Consider now what happens when player p implements an action $\pi^e = \pi' \neq \hat{\pi}$. Suppose that $\pi' = \pi^*$. Then, use the fact that $C < k_1$ and equality (10) to show that $U(\pi^*, \pi^*) < U(\pi^*, f(\pi^*))$. Hence, player b should choose $\pi = f(\pi^*)$. Since $f(\pi^*) \neq \pi^*$, it follows that $\pi^e = \pi^*$ is not a best response to $\pi = f(\pi^*)$. For the case in which π' is different from both π^* and $\hat{\pi}$, recall that the optimal choice for b is (i) $\pi = \pi^*$ or (ii) $\pi = f(\pi')$. If (i) is true, then the assumption that $\pi' \neq \pi^*$ implies that such a π' does not solve the problem of player p. If (ii) holds, then the fact that $\hat{\pi}$ is the only fixed point of f implies that $\pi' \neq f(\pi')$. Again, π' cannot be an optimal strategy for p.

Now, assume that $C > k_2$. Apply Lemma 3 again to show that $C > k_1$. Hence, (10) implies that $U(\pi^*, \pi^*) > U(\pi^*, f(\pi^*))$. So, if player p implements the action $\pi^e = \pi^*$, the central bank's best response is $\pi = \pi^*$. Thus, (π^*, π^*) is a stage Nash equilibrium. To show there is no other equilibrium, observe that optimality for player p implies that (π^*, π) is not an equilibrium for any $\pi \neq \pi^*$. Next, assume that player p plays $\pi^e = \pi' \neq \pi^*$. If $\pi' = \hat{\pi}$, the inequality $C > k_2$ implies that $U(\hat{\pi}, \pi^*) > U(\hat{\pi}, f(\hat{\pi}))$. Thus, b should play π^*, and as a consequence, $\hat{\pi}$ is not an optimal action for player p. If π' is different from both π^* and $\hat{\pi}$, the optimal action for the central bank will be π^* or $f(\pi')$. Once more, the facts that $\pi' \neq \pi^*$ and $\pi' \neq f(\pi')$ imply that such a π' is not an optimal strategy for player p.

Finally, consider the case in which $k_1 \leq C \leq k_2$. The inequality $C \geq k_1$ and (10) imply that $U(\pi^*, \pi^*) \geq U(\pi^*, f(\pi^*))$. Thus, if player p plays π^*, then player b can optimally play π^*. Therefore, (π^*, π^*) is a stage Nash equilibrium. Similarly, the inequality $C \leq k_2$ and (11) imply that $U(\hat{\pi}, f(\hat{\pi})) \geq U(\hat{\pi}, \pi^*)$. Hence, $(\hat{\pi}, \hat{\pi})$ is also a stage Nash equilibrium. To establish that there is no other equilibrium, observe that optimality for player p implies that (π^*, π) is not an equilibrium for any $\pi \neq \pi^*$, while $(\hat{\pi}, \pi)$ is not for any $\pi \neq \hat{\pi}$. Moreover, if player p plays $\pi' \notin \{\pi^*, \hat{\pi}\}$, then the optimal response for b is π^* or $f(\pi')$. Since $\pi' \neq \pi^*$ and $\pi' \neq f(\pi')$, it follows that π' cannot be an equilibrium strategy for p. ∎

Lemma 5 *If $\{x_t\}_{t=0}^{\infty}$ is an equilibrium outcome, then $\{x_t\}_{t=0}^{\infty}$ satisfies (14).*

Proof. Given a strategy σ for player b, the strategy of setting π_t^e equal to $\sigma_t(h^{t-1})$ will yield player p its maximum attainable payoff (which is 0) at every node of the game. Hence, $V(\pi_t^e, \pi_t) = 0$ in any equilibrium. Thus, (14) must hold in such an equilibrium. ∎

Proposition 6 *For $C \leq k_2$, a sequence $\{x_t\}_{t=0}^{\infty}$ belongs to $\mathcal{T}(C)$ if and only if it satisfies (14) and (16). For $C > k_2$, a sequence $\{x_t\}_{t=0}^{\infty}$ belongs to $\mathcal{T}(C)$ if and only if it satisfies (14) and (15).*

Proof. Suppose that $C \leq k_2$. Therefore, $(\hat{\pi}, \hat{\pi})$ is a stage Nash equilibrium. Hence, any sequence $\{x_t\}_{t=0}^{\infty}$ that satisfies (14) and (16) must belong to $\mathcal{T}(C)$. This establishes the "if" part. For the "only if" part, let $\{x_t\}_{t=0}^{\infty}$ be any element of $\mathcal{T}(C)$. Apply Lemma 5 to conclude that $\{x_t\}_{t=0}^{\infty}$ satisfies (14). Moreover, the fact that $\{x_t\}_{t=0}^{\infty}$ belongs to $\mathcal{T}(C)$ implies that this sequence must satisfy (16) or (15). However, $U(\hat{\pi}, \hat{\pi}) < U(\pi^*, \pi^*)$. Thus, if $\{x_t\}_{t=0}^{\infty}$ satisfies (15), then it must also satisfy (16). A similar argument can be applied to the case in which $C > k_2$. ∎

The last proposition establishes that expressions (14), (16), and (15) provide sufficient conditions for a sequence X to be an equilibrium outcome. A natural inquiry is whether an equilibrium outcome must satisfy the conditions in question. As shown in Lemma 5, the answer is yes for equality (14). Concerning (16) and (15), the question at hand is equivalent to asking whether the worst (from the point of view of the central bank) equilibrium outcome is equal to the worst stage Nash equilibrium. It turns out that, at least for a patient central bank, the answer is no. Indeed, let $\tilde{\pi}$ be any inflation rate in the interval $(\hat{\pi}, \pi_{\max})$. Regardless of the value of C, for δ sufficiently close to 1 there is an equilibrium outcome in which at date 0 both agents play $\tilde{\pi}$ and on subsequent dates they switch to the worst stage Nash equilibrium. Since $U(\tilde{\pi}, \tilde{\pi})$ is smaller than both $U(\hat{\pi}, \hat{\pi})$ and $U(\pi^*, \pi^*)$, no stage Nash equilibrium can be the worst equilibrium outcome of the repeated game.

Proposition 7 *The sequence X^* belongs to $\mathcal{T}(C)$ if and only if $C \geq k_0$.*

Proof. I start with the "if" part. Suppose that $C \geq k_0$. If $C \geq k_1$, then (π^*, π^*) is a stage Nash equilibrium, and as a consequence, X^* is an equilibrium outcome of the infinitely repeated game. If $C < k_1$, the fact that $C \geq k_0$ implies that

$$\begin{aligned} & C \geq (1-\delta)k_1 - \delta[W(\pi^*, \pi^*) - W(\hat{\pi}, \hat{\pi})] \\ & \qquad \Longrightarrow (1-\delta)C \geq (1-\delta)k_1 - \delta\{W(\pi^*, \pi^*) - [W(\hat{\pi}, \hat{\pi}) - C]\} \\ & \qquad\qquad \Longrightarrow C \geq k_1 - \frac{\delta}{1-\delta}\{U(\pi^*, \pi^*) - U(\hat{\pi}, \hat{\pi})\}. \end{aligned}$$

Combine the last inequality with (10) to conclude that

$$\begin{aligned} & C \geq W(\pi^*, f(\pi^*)) - W(\pi^*, \pi^*) - \frac{\delta}{1-\delta}\{U(\pi^*, \pi^*) - U(\hat{\pi}, \hat{\pi})\} \\ & \qquad \Longrightarrow W(\pi^*, \pi^*) + \frac{\delta}{1-\delta}U(\pi^*, \pi^*) \geq W(\pi^*, f(\pi^*)) - C + \frac{\delta}{1-\delta}U(\hat{\pi}, \hat{\pi}). \end{aligned}$$

However, $f(\pi^*) \neq \pi^*$. Therefore,

$$U(\pi^*, \pi^*) + \frac{\delta}{1-\delta}U(\pi^*, \pi^*) \geq U(\pi^*, f(\pi^*)) + \frac{\delta}{1-\delta}U(\hat{\pi}, \hat{\pi}). \tag{24}$$

Now, observe that the inequality $C < k_1$ implies that $U(\pi^*, \pi^*) < U(\pi^*, f(\pi^*))$. Thus, it is possible to apply Lemma 1 to show that $\max_{\pi \in \Pi} U(\pi^*, \pi) = U(\pi^*, f(\pi^*))$. Combine this equality with (24) to conclude that X^* satisfies (16). Hence, X^* is an equilibrium outcome.

Concerning the "only if" part, it suffices to show that if $C < k_0$, then $X^* \notin \mathcal{T}(C)$. Reasoning similar to that used to obtain (24) establishes that the reverse inequality holds strictly if $C < k_0$. Thus, player b can enhance its payoff by deviating to $f(\pi^*)$, and as a consequence, $X^* \notin \mathcal{T}(C)$. ∎

Proposition 8 *If $C_1 < C_2 \leq k_2$, then $\mathcal{T}(C_2) \cap \tilde{\mathbb{X}}^* \subseteq \mathcal{T}(C_1) \cap \tilde{\mathbb{X}}^*$.*

Proof. Let X be any element of $\mathcal{T}(C_2) \cap \tilde{\mathbb{X}}^*$. Thus, X satisfies (14). Furthermore, for any $X \in \tilde{\mathbb{X}}^*$, inequality (16) is equivalent to

$$\sum_{r=t+1}^{\infty} \delta^{r-t}[W(\pi_r^e, \pi_r) - W(\hat{\pi}, \hat{\pi})]$$
$$\geq \max\{W(\pi_t^e, \pi^*) + C, W(\pi_t^e, f(\pi_t^e)) + [1 - I(f(\pi_t^e))]C\} - W(\pi_t^e, \pi_t).$$

The right-hand side of this inequality is non-decreasing in C, while the left-hand side does not depend on the variable in question. Given that $C_1 < C_2$ and X satisfies this condition for $C = C_2$, it must be the case that the same is true for $C = C_1$. Therefore, $X \in \mathcal{T}(C_1) \cap \tilde{\mathbb{X}}^*$. ∎

Example 9 The sets $\mathcal{T}(C_1) \cap \tilde{\mathbb{X}}^*$ and $\mathcal{T}(C_2) \cap \tilde{\mathbb{X}}^*$ in Proposition 8 need not be equal. Let π' be an inflation rate belonging to the interval $(\pi^*, \hat{\pi})$ and X' be the sequence in which $\pi_t^e = \pi_t = \pi'$ for all t. Since $\pi^* < \pi' < \hat{\pi}$, (10), (11), (22), and (23) imply that

$$k_1 = \psi(\pi^*) < \psi(\pi') < \psi(\hat{\pi}) = k_2.$$

Furthermore, if $C > \psi(\pi')$, then $W(\pi', f(\pi')) - C < W(\pi', \pi^*)$. Hence, for any such C,

$$\max_{\pi \in \Pi}[W(\pi', \pi) - I(\pi)C] = W(\pi', \pi^*)$$

and

$$0 < W(\pi', f(\pi')) - W(\pi', \pi') < W(\pi', \pi^*) - W(\pi', \pi') + C.$$

Therefore,

$$\max_{\pi \in \Pi}[W(\pi', \pi) - I(\pi)C] - [W(\pi', \pi') - C] = W(\pi', \pi^*) - W(\pi', \pi') + C > 0. \tag{25}$$

Now, take a penalty $C_1 \in (\psi(\pi'), k_2)$ and a penalty $C_2 \in (C_1, k_2)$. Thus, the equality in (25) holds for $C = C_1$ and $C = C_2$. Next, define δ' so that

$$\frac{\delta'}{1 - \delta'}[W(\pi', \pi') - W(\hat{\pi}, \hat{\pi})] = W(\pi', \pi^*) - W(\pi', \pi') + C_1. \tag{26}$$

Combine the inequality $W(\pi', \pi') > W(\hat{\pi}, \hat{\pi})$ with (25) to conclude that δ' is well-defined and lies in the interval $(0, 1)$. Hence, if $\delta = \delta'$ and $C = C_1$, then X' satisfies (16) as an equality, which implies that $X' \in \mathcal{T}(C_1) \cap \tilde{\mathbb{X}}^*$. On the other hand,

$$W(\pi', \pi^*) - W(\pi', \pi') + C_1 < W(\pi', \pi^*) - W(\pi', \pi') + C_2.$$

Combine this inequality with (26) to conclude that X' does not satisfy (16) for $\delta = \delta'$ and $C = C_2$. As a consequence, $X' \notin \mathcal{T}(C_2) \cap \tilde{\mathbb{X}}^*$.

In the discussion that follows, $\mathbb{X}^*_\delta$ denotes the subset of $\mathbb{X}^*$ containing all sequences that satisfy (19) for every t.

Proposition 10 *If $C_1 < C_2 \leq k_2$, then $\mathcal{T}(C_1) \cap \mathbb{X}^*_\delta \subseteq \mathcal{T}(C_2) \cap \mathbb{X}^*_\delta$.*

Proof. Take a sequence X belonging to $\mathcal{T}(C_1) \cap \mathbb{X}^*_\delta$. Clearly, X satisfies (14). Condition (16) can be written as

$$\sum_{r=t+1}^{\infty} \delta^{r-t}[W(\pi^e_r, \pi_r) - W(\hat{\pi}, \hat{\pi})] + \left\{ \sum_{r=t+1}^{\infty} \delta^{r-t}[1 - I(\pi_r)] - I(\pi_t) \right\} C$$
$$\geq \max\{W(\pi^e_t, \pi^*), W(\pi^e_t, f(\pi^e_t)) - I(f(\pi^e_t))C\} - W(\pi^e_t, \pi_t).$$

Since $X \in \mathbb{X}^*_\delta$, (19) implies that the expression inside the large curly brackets is non-negative. Therefore, the left-hand side is non-decreasing in C, while the right-hand side is non-increasing in C. Since X satisfies the above inequality for $C = C_1$, X will also satisfy it for $C = C_2$. Hence, $X \in \mathcal{T}(C_2) \cap \mathbb{X}^*_\delta$. ∎

Example 11 The sets $\mathcal{T}(C_1) \cap X^*_\delta$ and $\mathcal{T}(C_2) \cap X^*_\delta$ in Proposition 10 need not be equal. For instance, let X' be the sequence in which $\pi^e_t = \pi_t = \hat{\pi}$ for t even and $\pi^e_t = \pi_t = \pi^*$ for t odd. Assume that $\delta = 0.62$. Therefore, $\delta/(1 - \delta^2) \cong 1.0071 > 1$. Thus, (19) holds, which implies that $X' \in \mathbb{X}^*_\delta$. Consider the expression

$$\sum_{r=t+1}^{\infty} \delta^{r-t} \{W(\pi^e_r, \pi_r) - W(\hat{\pi}, \hat{\pi}) + [1 - I(\pi_r)]C\} \tag{27}$$
$$\geq \max\{W(\pi^e_t, \pi^*), W(\pi^e_t, f(\pi^e_t)) - I(f(\pi^e_t))C\} - W(\pi^e_t, \pi_t) + I(\pi_t)C,$$

which is equivalent to (16). The term inside the curly brackets on the left-hand side is positive for r odd and equal to 0 for r even. Therefore, the left-hand side of the inequality is positive. Concerning the right-hand side, the inequality $C \leq k_2$ implies it is equal to 0 for t even. Thus, (27) holds for all even dates. For t odd, the right-hand side is equal to $\max\{0, k_1 - C\}$. Next, assume that $\alpha = 0.5$, $\gamma = \mu = \bar{y} = 1$, and $\pi^* = 0.01$. Hence, $\hat{\pi} = 0.5$ and $k_1 = 0.19208$. The table below contains the results of the numerical evaluation, for odd dates, of both sides of (27) for two distinct values of C.

Table 1
Numerical evaluation of the left- and
the right-hand sides of expression (27) for t odd

C	left-hand side	right-hand side
0.02	0.16853	0.17208
0.10	0.21849	0.09208

Thus, $X' \notin \mathcal{T}(0.02)$ and $X' \in \mathcal{T}(0.10)$; as a consequence, $\mathcal{T}(0.02) \cap \mathbb{X}^*_\delta \neq \mathcal{T}(0.10) \cap \mathbb{X}^*_\delta$.

Lemma 12 *If $C > k_2$, then $\hat{X} \notin \mathcal{T}(C)$.*

Proof. The inequality $C > k_2$ implies that $W(\hat{\pi}, \hat{\pi}) - C < W(\hat{\pi}, \pi^*)$. Thus,

$$\sum_{r=t}^{\infty} \delta^{r-t} U(\hat{\pi}, \hat{\pi}) = \frac{W(\hat{\pi}, \hat{\pi}) - C}{1-\delta} = W(\hat{\pi}, \hat{\pi}) - C + \delta \frac{W(\hat{\pi}, \hat{\pi}) - C}{1-\delta}$$
$$< W(\hat{\pi}, \pi^*) + \delta \frac{W(\pi^*, \pi^*)}{1-\delta} = U(\hat{\pi}, \pi^*) + \delta \frac{U(\pi^*, \pi^*)}{1-\delta} \leq \max_{\pi \in \Pi} U(\hat{\pi}, \pi) + \delta \frac{U(\pi^*, \pi^*)}{1-\delta}.$$

Therefore, $\hat{X}$ does not satisfy (15). As a consequence, $\hat{X} \notin \mathcal{T}(C)$. ∎

Lemma 13 *There exists a real number $\theta > 0$ such that $\mathcal{T}(C) \subset \mathcal{T}(k_2)$ for every $C \in (k_2, k_2 + \theta]$.*

Proof. Define θ so that

$$\theta = \frac{\delta}{1-\delta}[W(\pi^*, \pi^*) - W(\hat{\pi}, \hat{\pi}) + k_2]. \tag{28}$$

Now, suppose that C belongs to $(k_2, k_2 + \theta]$ and let X be any element of $\mathcal{T}(C)$. Condition (15) implies that

$$\sum_{r=t}^{\infty} \delta^{r-t} [W(\pi^e_r, \pi_r) - I(\pi_r)C] \geq \max_{\pi \in \Pi} [W(\pi^e_t, \pi) - I(\pi)C] + \delta \frac{W(\pi^*, \pi^*)}{1-\delta}.$$

As a consequence,

$$\sum_{r=t}^{\infty} \delta^{r-t}[W(\pi_r^e, \pi_r) - I(\pi_r)k_2] \geq \max_{\pi\in\Pi}[W(\pi_t^e, \pi) - I(\pi)(k_2+\theta)] + \delta \frac{W(\pi^*, \pi^*)}{1-\delta}.$$

Combine this inequality with (28) to conclude that

$$\sum_{r=t}^{\infty} \delta^{r-t}[W(\pi_r^e, \pi_r) - I(\pi_r)k_2] \geq \max_{\pi\in\Pi}[W(\pi_t^e, \pi) - I(\pi)(k_2+\theta)] + \theta + \delta \frac{W(\hat{\pi}, \hat{\pi}) - k_2}{1-\delta}. \quad (29)$$

On the other hand,

$$\max_{\pi\in\Pi}[W(\pi_t^e, \pi) - I(\pi)(k_2+\theta)] + \theta = \max_{\pi\in\Pi}\{W(\pi_t^e, \pi) - I(\pi)k_2 + [1 - I(\pi)]\theta\}$$
$$\implies \max_{\pi\in\Pi}[W(\pi_t^e, \pi) - I(\pi)(k_2+\theta)] + \theta \geq \max_{\pi\in\Pi}[W(\pi_t^e, \pi) - I(\pi)k_2].$$

Together, the last inequality and (29) imply that

$$\sum_{r=t}^{\infty} \delta^{r-t}[W(\pi_r^e, \pi_r) - I(\pi_r)k_2] \geq \max_{\pi\in\Pi}[W(\pi_t^e, \pi) - I(\pi)k_2] + \delta \frac{W(\hat{\pi}, \hat{\pi}) - k_2}{1-\delta}.$$

Therefore, X satisfies (16) when the penalty is equal to k_2. Since $X \in \mathcal{T}(C)$, X also satisfies (14). Thus, $X \in \mathcal{T}(k_2)$, from which it follows that $\mathcal{T}(C) \subseteq \mathcal{T}(k_2)$.

It remains to show that $\mathcal{T}(C) \neq \mathcal{T}(k_2)$. Recall that $\hat{X} \in \mathcal{T}(k_2)$, while Lemma 12 implies that $\hat{X} \notin \mathcal{T}(C)$. Hence, the two sets are indeed different. ∎

For future reference, define the function $R(\pi^e, \pi, C)$ according to

$$R(\pi^e, \pi, C) = \max\{W(\pi^e, \pi^*), W(\pi^e, f(\pi^e)) - C\} - [W(\pi^e, \pi) - I(\pi)C]. \quad (30)$$

Therefore, for sequences in which $\pi_t^e = \pi_t$ for all t, inequality (15) is equivalent to

$$\sum_{r=t+1}^{\infty} \delta^{r-t}\{[W(\pi_r, \pi_r) - I(\pi_r)C] - W(\pi^*, \pi^*)\} \geq R(\pi_t, \pi_t, C). \quad (31)$$

Lemma 14 *Suppose that $C > k_2$. Thus, $R(\pi^*, \pi^*, C) = 0$, $R(\pi, \pi, C) > 0$ for $\pi \neq \pi^*$, and $R(\pi, \pi, C)$ is non-decreasing in C.*

Proof. Take any $C > k_2$. Since $C > k_1$, $W(\pi^*, \pi^*) > W(\pi^*, f(\pi^*)) - C$. Combine this inequality with (30) to conclude that $R(\pi^*, \pi^*, C) = 0$. Next, observe that

$$\pi \neq \pi^* \implies R(\pi, \pi, C) = \max\{W(\pi, \pi^*) - W(\pi, \pi) + C, W(\pi, f(\pi)) - W(\pi, \pi)\}. \quad (32)$$

Since $W(\hat{\pi}, \pi^*) - W(\hat{\pi}, \hat{\pi}) + C = C - k_2$, $R(\hat{\pi}, \hat{\pi}, C) \geq C - k_2 > 0$. Therefore, $R(\pi, \pi, C) > 0$ if $\pi = \hat{\pi}$. On the other hand, if π is different from both $\hat{\pi}$ and π^*, then $f(\pi) \neq \pi$ and, as a consequence, $W(\pi, f(\pi)) - W(\pi, \pi) > 0$. I conclude again that $R(\pi, \pi, C) > 0$. Finally, an inspection of the equality in (32) establishes that $R(\pi, \pi, C)$ is non-decreasing in C whenever $\pi \neq \pi^*$. Since $R(\pi^*, \pi^*, C) = 0$ for all C, it must be the case that $R(\pi, \pi, C)$ is non-decreasing in C for all π. ∎

Proposition 15 *If $k_2 < C_1 < C_2$, then $\mathcal{T}(C_2) \subseteq \mathcal{T}(C_1)$.*

Proof. Consider two penalties C_1 and C_2 satisfying $k_2 < C_1 < C_2$. Let X be an element of $\mathcal{T}(C_2)$. Thus, X satisfies (31) for $C = C_2$. Now, observe that the left-hand side of (31) is non-increasing in C, while Lemma 14 implies that its right-hand side is non-decreasing in C. Therefore, X also satisfies (31) for $C = C_1$. Hence, $X \in \mathcal{T}(C_1)$, and as a consequence, $\mathcal{T}(C_2) \subseteq \mathcal{T}(C_1)$. ∎

It is worth pointing out that the sets $\mathcal{T}(C_1)$ and $\mathcal{T}(C_2)$ may be equal.

Proposition 16 *If $C > k_2$, then $\mathcal{T}(C) \subset \mathcal{T}(k_2)$.*

Proof. Let C be any penalty such that $C > k_2$. The previous proposition implies that $\mathcal{T}(C) \subseteq \mathcal{T}(C')$ holds for every $C' \in (k_2, C)$. By making C' sufficiently close to k_2, it is possible to apply Lemma 13 to conclude that $\mathcal{T}(C') \subset \mathcal{T}(k_2)$. Thus, $\mathcal{T}(C) \subset \mathcal{T}(k_2)$. ∎

Proposition 17 *For every $C > k_2$, there exists a positive number ε (that does not depend on δ) such that if a sequence $X \neq X^*$ has the property that $|\pi_t - \pi^*| < \varepsilon$ for all t, then $X \notin \mathcal{T}(C)$.*

Proof. Take a penalty $C > k_2$. The continuity of W implies that there exists an $\varepsilon > 0$ such that

$$|\pi - \pi^*| < \varepsilon \implies |W(\pi, \pi) - W(\pi^*, \pi^*)| < C. \tag{33}$$

Clearly, ε does not depend on δ. Now, observe that

$$\begin{aligned} |W(\pi, \pi) - W(\pi^*, \pi^*)| < C &\implies W(\pi, \pi) - C - W(\pi^*, \pi^*) < 0 \\ &\implies [W(\pi, \pi) - I(\pi)C] - W(\pi^*, \pi^*) \leq 0. \end{aligned} \tag{34}$$

Next, take a sequence $X \neq X^*$ with the property $|\pi_r - \pi^*| < \varepsilon$ for all r and let t be the first date on which $\pi_t \neq \pi^*$. Inequality (34) implies that the sum in the left-hand side of (31) is smaller than or equal to 0. On the other hand, the fact that $\pi_t \neq \pi^*$ combined with Lemma 14 implies that $R(\pi_t, \pi_t, C) > 0$. Thus, X does not satisfy (31); hence $X \notin \mathcal{T}(C)$. ∎

Proposition 18 *If $C > k_2$ and $C \geq k_3$, then $\mathcal{T}(C) = \{X^*\}$.*

Proof. Take any C that satisfies the stated conditions. Since $C > k_2$, $X^* \in \mathcal{T}(C)$. Next, take a sequence $X \neq X^*$ that satisfies (14). It suffices to show that X does not satisfy (31). Let t be the first date on which $\pi_t \neq \pi^*$. Apply Lemma 14 to conclude that $R(\pi_t, \pi_t, C) > 0$. Consider now the left-hand side of (31). If $\pi_r = \pi^*$, then the term inside the curly brackets is equal to 0. Next, suppose that $\pi_r \neq \pi^*$. Since $W(0,0) \geq W(\pi_r, \pi_r)$,

$$W(\pi_r, \pi_r) - W(\pi^*, \pi^*) \leq W(0,0) - W(\pi^*, \pi^*) = k_3 \implies W(\pi_r, \pi_r) - W(\pi^*, \pi^*) \leq C.$$

Hence, the term inside the curly brackets is nonpositive if $\pi_r \neq \pi^*$. As a consequence, the sum on the left-hand side of (31) is less than or equal to 0. Since $R(\pi_t, \pi_t, C) > 0$, it follows that X does not satisfy (31). ∎

As mentioned in the main body of the paper, it is not possible to say whether or not k_3 is larger than k_2 without further assumptions on the parameters. Indeed, a little algebra establishes that $k_2 = (\alpha^2 + \gamma)(\hat{\pi} - \pi^*)^2$ and $k_3 = (\pi^*)^2 \gamma$. Thus,

$$k_3 > k_2 \iff \pi^* > \frac{(\alpha^2 + \gamma)^{0.5}}{(\alpha^2 + \gamma)^{0.5} + \gamma^{0.5}} \hat{\pi}.$$

Next I investigate whether the converse of the last proposition is true. That is, I study whether the equality $\mathcal{T}(C) = \{X^*\}$ implies that $C > k_2$ and $C \geq k_3$. However, this is equivalent to studying if $C \leq k_2$ or $C < k_3$ implies that $\mathcal{T}(C) \neq \{X^*\}$. Since $\hat{X}$ will belong to $\mathcal{T}(C)$ whenever $C \leq k_2$, it remains to consider what happens when $C \in (k_2, k_3)$.

Suppose that $k_2 < k_3$ and take any C in the interval (k_2, k_3). Hence, (20) implies that $W(0,0) - W(\pi^*, \pi^*) - C > 0$. Now, let π_C be any inflation rate in the interval $(0, \pi^*)$ with the property that

$$W(\pi_C, \pi_C) - W(\pi^*, \pi^*) - C > 0 \tag{35}$$

and $\mathbb{X}(\pi_C)$ be the subset of all sequences in $\mathbb{X}$ such that $\pi_t^e = \pi_t$ and $\pi_t \in [0, \pi_C]$ for all t.

Proposition 19 *Suppose that $k_2 < k_3$. Hence, for every $C \in (k_2, k_3)$ there is a number $\delta_C \in (0,1)$ with the property that if $\delta \in [\delta_C, 1)$, then $\mathbb{X}(\pi_C) \subseteq \mathcal{T}(C)$.*

Proof. Define $\bar{R}$ according to

$$\bar{R} = \max_{\pi \in \Pi} \left[\max\{W(\pi, \pi^*) - W(\pi, \pi) + C, W(\pi, f(\pi)) - W(\pi, \pi)\} \right]. \tag{36}$$

Since the objective function is continuous and Π is compact, $\bar{R}$ is well defined. Furthermore, $W(\pi^*, f(\pi^*)) - W(\pi^*, \pi^*) > 0$, so $\bar{R} > 0$. Next, define δ_C so that

$$\delta_C = \frac{\bar{R}}{[W(\pi_C, \pi_C) - C - W(\pi^*, \pi^*)] + \bar{R}}.$$

Combine (35) with the fact $\bar{R} > 0$ to conclude that $\delta_C \in (0, 1)$. Now, take an $X \in \mathbb{X}(\pi_C)$ and let t be any date. Since $\pi_r \leq \pi_C < \pi^*$, $W(\pi_r, \pi_r) \geq W(\pi_C, \pi_C)$ and $I(\pi_r)C = C$. Hence,

$$W(\pi_r, \pi_r) - I(\pi_r)C - W(\pi^*, \pi^*) \geq W(\pi_C, \pi_C) - C - W(\pi^*, \pi^*).$$

Thus, if $\delta \in [\delta_C, 1)$, then

$$\sum_{r=t+1}^{\infty} \delta^{r-t}[W(\pi_r, \pi_r) - I(\pi_r)C - W(\pi^*, \pi^*)] \geq \frac{\delta_C}{1 - \delta_C}[W(\pi_C, \pi_C) - C - W(\pi^*, \pi^*)] = \bar{R}. \quad (37)$$

Now, compare the equality in (32) with the objective function in (36). Since $\pi_t \neq \pi^*$, it must be the case that $\bar{R} \geq R(\pi_t, \pi_t, C)$. Together with (37), this last inequality implies that X satisfies (31), and as a consequence, $X \in \mathcal{T}(C)$. ∎

One may wonder whether the assumption regarding δ can be dispensed with. As shown in the next proposition, the answer is no.

Proposition 20 *Suppose that $k_2 < k_3$. Hence, for every $C \in (k_2, k_3)$ there is a number $\delta'_C \in (0, 1)$ with the property that if $\delta \in (0, \delta'_C)$, then $\mathcal{T}(C) = \{X^*\}$.*

Proof. Given a penalty $C \in (k_2, k_3)$, take any ε as in the statement of Proposition 17 and let Π_ε be the set $\{\pi \in \Pi : |\pi - \pi^*| \geq \varepsilon\}$. Since $C < k_3$, (20) and (33) together imply that $0 \in \Pi_\varepsilon$. Therefore, Π_ε is not empty. Next, consider the problem of selecting $\pi \in \Pi_\varepsilon$ to minimize $R(\pi, \pi, C)$. Since $\pi^* \notin \Pi_\varepsilon$, the objective function is continuous in Π_ε. Thus, the compactness of this set implies that there is a solution π_ε. Clearly, $\pi_\varepsilon \neq \pi^*$; hence, Lemma 14 implies that $R(\pi_\varepsilon, \pi_\varepsilon, C) > 0$. Now, define δ'_C according to

$$\delta'_C = \frac{R(\pi_\varepsilon, \pi_\varepsilon, C)}{(k_3 - C) + R(\pi_\varepsilon, \pi_\varepsilon, C)}.$$

Observe that $\delta'_C \in (0, 1)$. Now, take a sequence $X \neq X^*$ that satisfies (14). Apply Proposition 17 to conclude that if $|\pi_t - \pi^*| < \varepsilon$ for all t, then $X \notin \mathcal{T}(C)$. Suppose now that

$|\pi_t - \pi^*| \geq \varepsilon$ for some date t. Since $W(\pi_r, \pi_r) \leq W(0,0)$,

$$W(\pi_r, \pi_r) - C - W(\pi^*, \pi^*) \leq W(0,0) - C - W(\pi^*, \pi^*) = k_3 - C$$
$$\Longrightarrow W(\pi_r, \pi_r) - I(\pi_r)C - W(\pi^*, \pi^*) \leq k_3 - C.$$

Hence, if $\delta \in (0, \delta'_C)$, then

$$\sum_{r=t+1}^{\infty} \delta^{r-t}[W(\pi_r, \pi_r) - I(\pi_r)C - W(\pi^*, \pi^*)] < \frac{\delta'_C}{1-\delta'_C}(k_3 - C) = R(\pi_\varepsilon, \pi_\varepsilon, C).$$

Furthermore, $\pi_t \in \Pi_\varepsilon$, so $R(\pi_\varepsilon, \pi_\varepsilon, C) \leq R(\pi_t, \pi_t, C)$. Therefore, X does not satisfy (31) and this implies that $X \notin \mathcal{T}(C)$. ∎

I close this appendix by pointing out that the critical discount rates δ_C and δ'_C are uniform over $\mathbb{X}$. That is, they do not depend on X.

Appendix B: a high target rate

In this appendix, I study what happens when $\pi^* \geq \hat{\pi}$. I divide the analysis into two cases: (i) $\pi^* = \hat{\pi}$ and (ii) $\pi^* > \hat{\pi}$. The main findings of the paper remain true in both contexts.

Case (i)

Since $\pi^* = \hat{\pi}$, the stage game has a single Nash equilibrium regardless of the value of C. Arguments similar to those in Appendix A can be used to establish the results that follow. To facilitate the comparison, I label these new results with the corresponding numbers used in the baseline case, but followed by the prime symbol.

Proposition 6′ A sequence $\{x_t\}_{t=0}^{\infty}$ belongs to $\mathcal{T}(C)$ if and only if it satisfies (14) and (15).

Proposition 7′ The sequence X^* belongs to $\mathcal{T}(C)$.

Proposition 15′ If $C_1 < C_2$, then $\mathcal{T}(C_2) \subseteq \mathcal{T}(C_1)$.

Proposition 17′ For every C, there exists a positive number ε (that does not depend on δ) such that if a sequence $X \neq X^*$ has the property that $|\pi_t - \pi^*| < \varepsilon$ for all t, then $X \notin \mathcal{T}(C)$.

Proposition 18′ If $C \geq k_3$, then $\mathcal{T}(C) = \{X^*\}$.

Proposition 19′ For every $C < k_3$, there is a number $\delta_C \in (0,1)$ with the property that if $\delta \in [\delta_C, 1)$, then $\mathbb{X}(\pi_C) \subseteq \mathcal{T}(C)$.

Proposition 20′ For every $C < k_3$, there is a number $\delta'_C \in (0, 1)$ with the property that if $\delta \in (0, \delta'_C)$, then $\mathcal{T}(C) = \{X^*\}$.

Observe that if $\pi^* = \hat{\pi}$, then $k_1 = k_2 = 0$. Consequently, there are no counterparts for Propositions 8, 10, and 16 here.

Case (ii)

In this context, $\pi^* > \hat{\pi}$. Small variations of the argument used to establish Lemma 3 suffice to show that the same result holds here. Therefore, the analysis of the stage game is not changed. I now turn to the study of the infinitely repeated game.

It is necessary to introduce a new parameter allowing for the identification of the worst stage Nash equilibrium when $C \in [k_1, k_2]$. Define k_4 so that

$$k_4 = W(\hat{\pi}, \hat{\pi}) - W(\pi^*, \pi^*).$$

Therefore, $U(\pi^*, \pi^*) > U(\hat{\pi}, \hat{\pi})$ if and only if $C > k_4$.[1]

It is a relatively simple exercise to show that $k_4 > k_1$. However, it is not possible to say whether or not $k_4 \geq k_2$. Thus, the analysis has to be divided into two subcases: (a) $k_4 \geq k_2$ and (b) $k_4 < k_2$.

Subcase (ii.a)

Since $k_4 \geq k_2$, (π^*, π^*) is the worst stage Nash equilibrium for every C in $[k_1, k_2]$. Therefore, it is possible to establish the result that follows. Similarly to the previous case, the propositions are labeled with the same numbers as their counterparts in Appendix A, but now followed by a double prime and the letter a.

Proposition 6″a For $C < k_1$, a sequence $\{x_t\}_{t=0}^{\infty}$ belongs to $\mathcal{T}(C)$ if and only if it satisfies (14) and (16). For $C \geq k_1$, a sequence $\{x_t\}_{t=0}^{\infty}$ belongs to $\mathcal{T}(C)$ if and only if it satisfies (14) and (15).

Consider now Proposition 7. In this context, k_0 is strictly positive; however, it is not possible to say whether or not $k_0 < k_1$. For this reason, it is necessary to introduce an additional condition.

Proposition 7″a Suppose that $C < k_1$. Thus, the sequence X^* belongs to $\mathcal{T}(C)$ if and only if $C \geq k_0$.

[1] This step was not necessary in the case in which $\pi^* < \hat{\pi}$ because there $(\hat{\pi}, \hat{\pi})$ is the worst stage equilibrium for all C in the interval $[k_1, k_2]$.

The case in which $C \geq k_1$ is immediate, since (π^*, π^*) is a stage Nash equilibrium whenever the last inequality holds. Next I present counterparts to Propositions 8, 10, and 15.

Proposition 8″a If $C_1 < C_2 < k_1$, then $\mathcal{T}(C_2) \cap \tilde{\mathbb{X}}^* \subseteq \mathcal{T}(C_1) \cap \tilde{\mathbb{X}}^*$.

Proposition 10″a If $C_1 < C_2 < k_1$, then $\mathcal{T}(C_1) \cap \mathbb{X}^*_\delta \subseteq \mathcal{T}(C_2) \cap \mathbb{X}^*_\delta$.

Proposition 15″a If $k_1 \leq C_1 < C_2$, then $\mathcal{T}(C_2) \subseteq \mathcal{T}(C_1)$.

It is not possible to obtain a result similar to Proposition 16. Propositions 17, 18, 19, and 20 hold exactly as stated in Appendix A.

Subcase (ii.b)

In this context, $k_4 < k_2$. Thus, the worst stage Nash equilibrium is (π^*, π^*) for $C \in [k_1, k_4]$ and $(\hat{\pi}, \hat{\pi})$ for $C \in (k_4, k_2]$. Below I present counterparts, identified by the extra label b, to Propositions 6, 7, 8, 10, and 15.

Proposition 6″b For $C < k_1$ or $C \in (k_4, k_2]$, a sequence $\{x_t\}_{t=0}^\infty$ belongs to $\mathcal{T}(C)$ if and only if it satisfies (14) and (16). For $C \in [k_1, k_4]$ or $C > k_2$, a sequence $\{x_t\}_{t=0}^\infty$ belongs to $\mathcal{T}(C)$ if and only if it satisfies (14) and (15).

Proposition 7″b Suppose that $C < k_1$. Thus, the sequence X^* belongs to $\mathcal{T}(C)$ if and only if $C \geq k_0$.

Proposition 8″b If $C_1 < C_2 < k_1$ or $k_4 < C_1 < C_2 \leq k_2$, then $\mathcal{T}(C_2) \cap \tilde{\mathbb{X}}^* \subseteq \mathcal{T}(C_1) \cap \tilde{\mathbb{X}}^*$.

Proposition 10″b If $C_1 < C_2 < k_1$ or $k_4 < C_1 < C_2 \leq k_2$, then $\mathcal{T}(C_1) \cap \mathbb{X}^*_\delta \subseteq \mathcal{T}(C_2) \cap \mathbb{X}^*_\delta$.

Proposition 15″b If $k_1 \leq C_1 < C_2 \leq k_4$ or $k_2 < C_1 < C_2$, then $\mathcal{T}(C_2) \subseteq \mathcal{T}(C_1)$.

Propositions 16, 17, 18, 19, and 20 hold exactly as stated in Appendix A.