Optimal Exchange Rate Policy and Business Cycles

Alexandre B. Cunha
Faculdades Ibmec
Av Rio Branco 108, 5^o andar
Rio de Janeiro, RJ
Brazil 20040-001
abcunha@ibmecrj.br

February 12, 2002

Abstract

Implementation and collapse of exchange rate pegging schemes are recurrent events. A currency crisis (pegging) is often followed by an economic downturn (boom). This essay explains why a benevolent Central Bank should pursue a monetary policy that leads to those recurrent currency crises and subsequent periods of pegging. It is shown that the optimal policy induces a competitive equilibrium that displays a boom in periods of below average devaluation and a recession in periods of above average devaluation. A currency crisis (pegging) can be understood as an optimal policy answer to a recession (boom).

keywords: exchange rate, business cycles.
JEL: E32, F31, F41.

1 Introduction

In June 1975 a 16% currency devaluation took place in Argentina. Up to the previous month the nominal exchange rate was fixed. The GDP, which had grown 8.5% in 1974, fell 2.7% in 1975. A reverse episode happened in 1991. The Argentinean currency devaluated 68.7% in January and only 5.9% in February. The GDP rose 8.9% in 1991, against the modest increase of 0.1% in 1990.

Instead of constituting an isolated episode, the Argentinean experience illustrates a general pattern. Implementation and collapse of exchange rate pegging schemes are recurrent events. Several countries have experienced episodes in which pegging schemes were either changed or temporarily abandoned and later reinstated. Currency crises are frequently followed by a fall below the trend in output and consumption and a real depreciation of the domestic currency. The reverse facts plus a deterioration of the current account often accompany a pegging.

The first goal of this paper is to explain why governments optimally choose to pursue actions that lead to a recurrent implementation of pegging policies and their subsequent breakdown. The second goal is to reproduce the corresponding business cycles regularities.

The environment studied in this paper is an infinite horizon stochastic one. The model is a cash-credit two sector (tradable and non tradable) small open economy. Consumers face a cash-in-advance constraint on fraction of their purchases of non tradables. Tax rates on labor income, government consumption and few other variables are stochastic processes.

The essay builds on Lucas and Stokey's [12] seminal work on optimal monetary and fiscal policy. The problem of selecting the optimal monetary and exchange rate policy when the fiscal policy is exogenous is addressed.

The optimal devaluation rate is a non constant function of the economy's state. As this state changes, the devaluation rate oscillates. That will lead to the implementation and collapse of exchange rate pegging policies. This policy switch can happen infinitely often. Most (if not all) of the previous research on currency crisis can explain at most one devaluation episode.

In periods of fiscal deficits the devaluation is higher than in times of surpluses. The intuition is simple. Whenever public deficit is relatively high, the optimal policy will prescribe a combination of higher taxation and debt issuing. If the government is not allowed to change the fiscal variables, the only possible way to raise additional tax revenue is through inflation. A higher inflation level will determine a higher rate of devaluation of the domestic currency. A positive technological shock that leads to an output rise will reduce the fiscal deficit as a fraction of GDP. The previous reasoning shows that currency devaluation and technological shocks are negatively correlated.

Consider the business cycle facts associated to a pegging. In response to shocks that decrease the fiscal deficit and increase the productivity, the optimal policy prescribes a decrease in the devaluation rate. Higher productivity leads to higher output. The combination of lower devaluation rate and higher output generates an income effect. People increase their consumption sufficiently to induce a current account deficit. The higher demand for tradable goods can be partially offset by imports. Non tradables become relatively more expensive. Thus, the real exchange rate appreciates. In a similar fashion, shocks that increase the fiscal deficit and reduce productivity lead to a higher devaluation and induce the empirical regularities associated with currency crises.

There exists a traditional wisdom that a currency crisis triggers a recession. In this essay, a large devaluation does not cause a recession. The government optimally chooses to devaluate the currency when the economy hits a bad state. This finding has a striking policy implication. Given that the economy is facing a recession, a devaluation is an optimal answer. Any policy that prevents or just postpones the devaluation will lead to welfare losses.

Exchange rate devaluations are often viewed as a consequence of time consistency problems, as in Obstfeld [16] and Giavazzi and Pagano [7]. In this essay, devaluations are fully anticipated and are optimal choices for a government that can credibly commit to a policy.

Obstfeld and Rogoff [17] advocated the use of models with solid micro foundations to study exchange rate policy. Obstfeld [16] pointed out the relevance of understanding how the exchange rate policy is selected. Today, there exist several papers on open economy macro with micro foundations. However, few of them study the selection of the exchange rate policy. This paper attempts to answer the question raised by Obstfeld [16] using the approach advocated by Obstfeld and Rogoff [17].

Several studies on currency crises take an exchange rate pegging as given and explain why a currency crisis must happen, as in Krugman [10]. Other essays explain why a government chooses to devaluate, even if pegging is still possible, as in Obstfeld [16] and Giavazzi and Pagano [7]. Nevertheless, no essay is aimed at explaining why a pegging is ever introduced. This essay innovates by adopting a unified framework to explain both pegging episodes and currency crises.

So far, the research in the field of exchange rate based stabilization has taken the exchange rate policy as exogenous. Rebelo [18] states that it is important to understand the timing of the stabilization. This paper shows that a stabilization may be an optimal answer to a fiscal contraction. This important step is taken in the context of a model that replicates several of the stylized facts listed by Mendoza and Uribe [13], Rebelo [18], and Rebelo and Végh [19].

The paper is organized as follows. A summary of the empirical evidence on exchange rate pegging and currency crisis is presented in section 2. The model is described in section 3. Section 4 is devoted to characterization and examples of competitive equilibrium. The problem of selecting policies that lead to the best competitive equilibrium is studied in section 5, along with the properties of this efficient outcome. Section 6 concludes. Technical details are presented in the appendix.

2 The Empirical Regularities

This section summarizes some previous empirical research on currency pegging and crisis. The available evidence points towards three major sets of empirical regularities:

1. The implementation and collapse of pegging policies are worldwide recurrent events.

2. An exchange rate pegging is often accompanied by:

   (a) an increase, relative to trend, in consumption and output;

   (b) a deterioration in the current account;

   (c) an increase in real wages;

   (d) a real appreciation of the exchange rate;

   (e) a reduction in the fiscal deficit.

3. A currency crisis is often followed by:

   (a) a decrease, relative to trend, in consumption and output;

   (b) a depreciation of the real exchange rate.

The empirical work of Frankel and Rose [6], Klein and Marion [9], and Milesi-Ferretti and Razin [14] documents item 1. Kiguel and Liviatan [8] and Végh [21] provide evidence in support of item 2. Milesi-Ferretti and Razin [14] do the same for item 3.

3 The Economy

Consider a small country populated by a continuum of identical infinitely lived households with Lebesgue measure one and a government. A household is composed by a shopper and a worker, who is endowed with one unit of time.

The country produces two non tradable goods. The first is consumed by households (c_1^N). The second is consumed by households (c_2^N) and government (g^N). The country also produce a tradable good, which is consumed by households (c^T) and a government (g^T). This last good can also be exported (x) or imported ($-x$).

Markets operate in a particular way. At a first stage of each date t, a spot market for goods and labor services operates. At a second stage, after the market for goods and labor service closes, a security and currency market operate.

A domestic currency M circulate in this economy. Two types of securities are traded: a claim B to one unit of M and a claim B^* to one unit of some foreign currency. Both claims have maturity of one period. Foreigners do not sell or buy claims to the domestic currency. Government and residents can purchase and/or sell the claims B^* at an exogenous price, in terms of the foreign currency, q_t^*.

Workers cannot sell their services outside the country. Shoppers face a cash-in-advance constraint. The purchases of c_1^N must be paid for with the domestic currency. Except for the purchases of that good, all other transactions are liquidated during the security and currency trading session. The date t price, in terms of the foreign currency, of the tradable good is exogenous and equal to p_t^*.

Technology is described by $0 \leq y^T \leq \theta^T (l^T)^{\alpha^T}$ and $0 \leq y^N \leq \theta^N (l^N)^{\alpha^N}$, where y^T is the tradable output and l^T is the amount of labor allocated to the production of that good. Similar meanings are assigned to y^N and l^N. Both α^T and α^N lie in the set $(0, 1]$.

At each date t labor income is taxed at a proportional rate τ_t. Let $s_t = (\theta_t^T, \theta_t^N, g_t^T, g_t^N, \tau_t, p_t^*, q_t^*)$. The sequence $\{s_t\}_{t=0}^{\infty}$ is a stochastic process on some probability space $(\Omega, \mathcal{F}, P)$. Each s_t has a support contained in the finite set $S = \Theta^T \times \Theta^N \times G^T \times G^N \times \Upsilon \times P^* \times Q^*$. These sets satisfy

$\Theta^T \subset \mathbb{R}_{++}$, $\Theta^N \subset \mathbb{R}_{++}$, $G^T \subset \mathbb{R}_+$, $G^N \subset \mathbb{R}_{++}$, $\Upsilon \subset [0,1)$, $P^* \subset \mathbb{R}_{++}$, and $Q^* \subset (0,1)$. The object s^t stands for a history $(s_0, ..., s_t)$ of events and $s^\infty = (s_0, s_1, ...)$.

For a given t, S^t denotes the set of all possible histories s^t and S^∞ is the set of all possible s^∞. For a given s^t in S^t, $\mu(s^t)$ denotes the probability that this particular s^t will be realized. The realization of s_t is known at the beginning of date t. If $k \leq t$, $\mu(s^t|s^k)$ denotes the conditional probability of s^t given s^k; $S^t(s^k)$ is the set of all $s^t \in S^t$ such that the first k events in s^t are equal to s^k. In other words, $S^t(s^k)$ is the set of all possible continuations of the history s^k up to date t. Whenever there is no danger of confusion, $S^t(s^k)$ will be denoted by S^t_k. As usual, $\{[f(s^t)]_{s^t \in S^t}\}_{t=0}^\infty$ is a history contingent sequence and $\|f\|_\infty = \sup_{t,s^t} |f(s^t)|$.

Each good is produced by a single competitive firm. Let $l(s^t)$ denote the amount of labor supplied by each household at date t if the history s^t occurs. Other variables indexed by s^t have analogous meaning. Feasibility requires

$$l^T(s^t) + l^N(s^t) = l(s^t) \leq 1 \ , \ c_1^N(s^t) + c_2^N(s^t) + g_t^N = \theta_t^N [l^N(s^t)]^{\alpha^N} \ ,$$

$$c^T(s^t) + g_t^T + x(s^t) = \theta_t^T [l^T(s^t)]^{\alpha^T} \ . \tag{1}$$

The government finances the sequence $\{g_t^T, g_t^N\}_{t=0}^\infty$ by issuing and withdrawing the domestic currency; by issuing and redeeming claims B of maturity of one period to one unit of the domestic currency; by purchasing and selling B^*; by levying lump-sum taxes on profits; and taxing labor income. The government budget constraint is

$$E(s^t)p_t^* g_t^T + p^N(s^t) g_t^N + B(s^{t-1}) + E(s^t) q_t^* B_G^*(s^t) + M(s^{t-1}) =$$

$$\tau_t w(s^t) l(s^t) + q(s^t) B(s^t) + E(s^t) B_G^*(s^{t-1}) + M(s^t) + \psi^T(s^t) + \psi^N(s^t) \ , \tag{2}$$

where $p^N(s^t)$, $w(s^t)$ and $q(s^t)$ are the respective date t monetary prices (in terms of the domestic currency) of the non tradable good, labor services and the domestic claim; $E(s^t)$ is the nominal exchange rate; $B_G^*(s^t)$ stands for the foreign assets held by the government at the end of date t; $M(s^t)$ and $B(s^t)$ are the amount of domestic currency and public debt held by the households at the end of date t. $\psi^T(s^t)$ and $\psi^N(s^t)$ are the date t profits. All those variables are conditional on the history of events. A negative value for $B_G^*(s^t)$ means that the government is borrowing abroad, while a negative value for $B(s^t)$ means that the government is lending to domestic residents. At $t = 0$ the government holds an initial amounts $\bar{B}_G^*$ of foreign assets. To avoid Ponzi schemes, a standard boundedness constraint $\|B_G^*/p^*\|_\infty \leq A < \infty$ is imposed on government foreign assets.

It is usually assumed that profits are appropriated by households. Sticking to that procedure would make the incoming implementability constraint (9) to depend on government's consumption. That would make the computation of the optimal policies more expensive. Thus, for simplicity it was assumed that government taxes profits away. That assumption is not essential to the results.

The function $u : \mathbb{R}_+^3 \times [0,1] \to \mathbb{R} \cup \{-\infty\}$,

$$u(c^T, c_1^N, c_2^N, l) = \frac{\left[(c^T)^{\gamma^T} (c_1^N)^{\gamma_1} (c_2^N)^{\gamma_2} (1-l)^{\gamma^l}\right]^{1-\sigma}}{1-\sigma} \ , \tag{3}$$

is the typical household period utility function. The γ's are positive and add up to 1 and $\sigma \geq 0$. Intertemporal preferences are described by

$$\sum_{t=0}^\infty \sum_{s^t \in S^t} \beta^t \mu(s^t) u\left(c^T(s^t), c_1^N(s^t), c_2^N(s^t), l(s^t)\right) \ , \tag{4}$$

where $\beta \in (0,1)$. The date t budget constraint of the typical household is

$$E(s^t) p_t^* c^T(s^t) + p^N(s^t)[c_1^N(s^t) + c_2^N(s^t)] + q(s^t) B(s^t) + E(s^t) q_t^* B_H^*(s^t) + M(s^t) \leq$$

$$(1-\tau_t)w(s^t)l(s^t)+B(s^{t-1})+E(s^t)B_H^*(s^{t-1})+M(s^{t-1})\ , \tag{5}$$

where $B_H^*(s^t)$ stands for the foreign assets held by the household at the end of date t if history s^t occurs. The constraint $\|B/p^N\|_\infty, \|B_H^*/p^*\|_\infty \leq A < \infty$ prevents Ponzi games. People face the cash-in-advance constraint

$$p^N(s^t)c_1^N(s^t) \leq M(s^{t-1})\ . \tag{6}$$

Given initial cash and bond holdings $(\bar{M}, \bar{B}, \bar{B}_H^*)$, a household chooses a state contingent sequence $\{[c^T(s^t), c_1^N(s^t), c_2^N(s^t), l(s^t), B(s^t), B_H^*(s^t)]_{s^t\in S^t}\}_{t=0}^\infty$ to maximize (4) subject to the constraints (5), (6), and $l(s^t) \leq 1$. Except for $B(s^t)$ and $B_H^*(s^t)$, all those variables are constrained to be non-negative. An additional boundedness condition $\|c^T\|_\infty, \|c_1^N\|_\infty, \|c_2^N\|_\infty, \|l\|_\infty, \|M/p^N\|_\infty < \infty$ is imposed on the consumer problem.

4 Competitive Equilibrium

A monopolist firm cannot target both price and quantity. Similarly, the government cannot target domestic debt, money supply, foreign assets, interest rates, price level, and exchange rate. Without loss of generality, it will be assumed that the government target the last three variables.

A history contingent date t policy $(E(s^t), p^N(s^t), w(s^t), q(s^t))$ is denoted by $\varphi(s^t)$. A *policy* is an object $\varphi = \{[\varphi(s^t)]_{s^t\in S^t}\}_{t=0}^\infty$. Date t history contingent asset holdings $(M(s^t), B(s^t), B_H^*(s^t), B_G^*(s^t))$ and allocations $(l^T(s^t), l^N(s^t), x(s^t), c^T(s^t), c_1^N(s^t), c_2^N(s^t), l(s^t))$ are denoted, respectively, by $\zeta(s^t)$ and $\chi(s^t)$. Additionally, $\chi = \{[\chi(s^t)]_{s^t\in S^t}\}_{t=0}^\infty$ and $\zeta = \{[\zeta(s^t)]_{s^t\in S^t}\}_{t=0}^\infty$.

Definition 1 *A* competitive equilibrium *is an object (φ, χ, ζ) satisfying: (i) given φ, (χ, ζ) provides a solution for the household problem; (ii) given $\varphi(s^t)$, $l^T(s^t)$ and $l^N(s^t)$ maximize the respective firm's profit; (iii) (1) and (2) hold.*

Two points in the above definition should be emphasized. Item (ii) is equivalent to $w(s^t) = p^N(s^t)\alpha^N\theta_t^N[l^N(s^t)]^{\alpha^N-1}$ and $w(s^t) = E(s^t)p_t^*\alpha^T\theta_t^T[l^T(s^t)]^{\alpha^T-1}$. Adding the identities $\psi^N(s^t) + w(s^t)l^N(s^t) = p^N(s^t)[\ c_1^N(s^t) + c_2^N(s^t) + g_t^N]$ and $\psi^T(s^t) + w(s^t)l^T(s^t) = E(s^t)p_t^*[c^T(s^t) + g_t^T + x(s^t)]$ to (2) and (5) taken as equality, one obtains

$$p_t^*x(s^t) + B_G^*(s^{t-1}) + B_H^*(s^{t-1}) - q_t^*B_G^*(s^t) - q_t^*B_H^*(s^t) = 0\ , \tag{7}$$

which is the balance-of-payments identity for this economy. So, it is not necessary to spell this condition out when defining competitive equilibrium.

The definition of competitive equilibrium does not place bounds on inflation (throughout this essay the term inflation will apply to the rate of increase in p^N). For future reference, it is convenient to spell out a particular boundedness requirement.

Definition 2 *A competitive equilibrium (φ, χ, ζ) is of* bounded inflation *if*

$$\exists\varepsilon > 0 : \varepsilon \leq \frac{p^N(s^t, s_{t+1})}{p^N(s^t)} \leq \frac{1}{\varepsilon}\ ,\ \forall s^{t+1} \in S^{t+1}, \forall t\ . \tag{8}$$

The above condition prevents prices from increasing or decreasing "too much" in a single period. It is only a technical condition. Its relevance will become clear in the next section.

A set of competitive equilibrium allocations will be characterized next. That will reduce the problem of selecting an efficient policy, which will be discussed in section 5, to a standard constrained maximization problem.

To simplify the notation, $u(s^t)$, $u_T(s^t)$, $u_1(s^t)$, $u_2(s^t)$, and $u_l(s^t)$ will denote, respectively, the value of u and its partial derivatives $\partial u/\partial c^T$, $\partial u/\partial c_1^N$, $\partial u/\partial c_2^N$, and $\partial u/\partial l$ evaluated at the point

$(c^T(s^t), c_1^N(s^t), c_2^N(s^t), l(s^t))$. The sum $u_T(s^t)c^T(s^t) + u_1(s^t)c_1^N(s^t) + u_2(s^t)c_2^N(s^t) + u_l(s^t)l(s^t)$ will be denoted by $W(s^t)$.

There exist six conditions with obvious economic meaning that must hold in any competitive equilibrium. A trivial condition is (1). The second is

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \mu(s^t) W(s^t) = u_1(s^0) c_1^N(s^0) + u_2(s^0) \left[\frac{\bar{B}}{p^N(s^0)} + \frac{\bar{M}}{p^N(s^0)} - c_1^N(s^0) \right] + u_T(s^0) \frac{\bar{B}_H^*}{p_0^*} , \quad (9)$$

which consolidates all date t budget constraints of the households. The third is a balance-of-payment constraint

$$-\sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \mu(s^t) u_T(s^t) x(s^t) = u_T(s^0) \frac{\bar{B}_H^* + \bar{B}_G^*}{p_0^*} , \quad (10)$$

which requires imports to be financed by country's initial wealth. The fourth requirement, ensuring that people's marginal rate of substitution is consistent with the q_t^* and p_t^*, is

$$q_t^* \frac{\mu(s^t) u_T(s^t)}{p_t^*} = \beta \sum_{s_{t+1} \in S} \frac{\mu(s^t, s_{t+1}) u_T(s^t, s_{t+1})}{p_{t+1}^*} . \quad (11)$$

The fifth constraint is that households' marginal rate of substitution between tradables and non tradables must match the marginal rate of transformation between those types of goods, i.e.,

$$\frac{u_T(s^t)}{u_2(s^t)} = \frac{\alpha^N \theta_t^N [l^T(s^t)]^{1-\alpha^T}}{\alpha^T \theta_t^T [l^N(s^t)]^{1-\alpha^N}} . \quad (12)$$

This equation is also an implementability condition for the real exchange rate $\frac{E(s^t) p_t^*}{p^N(s^t)}$. The sixth

$$(1 - \tau_t) \frac{\alpha^N \theta_t^N}{[l^N(s^t)]^{1-\alpha^N}} = -\frac{u_l(s^t)}{u_2(s^t)} , \quad (13)$$

is an implementability constraint for labor income taxation.

The above constraints are not enough to characterize a competitive equilibrium. Other conditions have to be imposed. The inequalities

$$p^N(s^0) c_1^N(s^0) \leq \bar{M} \quad (14)$$

$$u_2(s^t) \leq u_1(s^t) \quad (15)$$

ensure that cash-in-advance constraints hold. An implementability constraint for a transversality condition is

$$\lim_{t \to \infty} \beta^t \sum_{s^t \in S_k^t} \mu(s^t) u_1(s^t) c_1^N(s^t) = 0 . \quad (16)$$

The boundedness of foreign debt requires

$$\sup_k \sup_{s^k \in S^k} \frac{1}{u_T(s^k)} \left| \sum_{t=k}^{\infty} \sum_{s^t \in S^t} \beta^{t-k} \mu(s^t | s^k) u_T(s^t) x(s^t) \right| < \infty . \quad (17)$$

A similar constraint is required to ensure that $\left\| B/p^N \right\|_\infty < \infty$. However, it is not possible to characterize that condition for all competitive equilibria. Nevertheless, it is possible to do so for all equilibria with bounded inflation. If the inflation is bounded, it is enough to require

$$\sup_k \sup_{s^k \in S^k} \frac{1}{u_2(s^k)} \left| \sum_{t=k}^{\infty} \sum_{s^t \in S^t} \beta^{t-k} \mu(s^t | s^k) W(s^t) - u_1(s^k) c_1^N(s^k) \right| < \infty . \quad (18)$$

Inflation is bounded if there exists a positive ε such that

$$\varepsilon \leq \frac{\beta}{\mu(s^t)u_2(s^t)} \sum_{\hat{s}_{t+1}\in S} \mu(s^t,\hat{s}_{t+1})u_1(s^t,\hat{s}_{t+1})\frac{c_1^N(s^t,\hat{s}_{t+1})}{c_1^N(s^t,s_{t+1})} \leq \frac{1}{\varepsilon} \ . \tag{19}$$

As in definition 2, ε does not depend on the histories.

Proposition 1 (a set of competitive equilibria) *Let $\bar{M} > 0$. An array χ and a price $p^N(s^0) > 0$ satisfy (1) and (9)-(19) if and only if they are components of a competitive equilibrium (φ, χ, ζ) of bounded inflation.*
Proof. *See appendix.* ■

The proof of proposition 1 is a long but straightforward exercise. It is enough to modify the techniques discussed by other authors (for instance, Chari and Kehoe [3] or Lucas and Stokey [12]) to the model discussed in this essay.

The above set of constraints do not include an implementability condition for the government budget constraint. However, there are implementability conditions for people's budget constraint, resource constraints and balance-of-payments. The government budget constraint is a linear combination of those other constraints.

Although fiscal variables are exogenous, the government can pursue several distinct policies. To clarify this point, consider the simple case in which the government has no source of revenue but inflation, at date zero the government has no net debt, and government consumption is always positive. The government can balance its lifetime budget with a constant inflation rate and borrow abroad to finance temporary imbalances. It is also possible to balance the budget period by period solely with inflation tax. In this case the inflation does not need to be constant. Different policies will induce distinct competitive equilibria. Some examples in the next section will reemphasize this point.

5 Ramsey Equilibrium

One of the goals of this paper is to explain why a government will select some exchange rate policy. The concept of competitive equilibrium alone does not help to answer this type of question. One must consider a game in which the government is an active player.

A specific game will be considered. At date zero, before markets open, the government announces that will follow a policy φ. That policy cannot be changed in future dates. Then, economic agents will trade. The government is benevolent and will choose φ to maximize (4). A standard equilibrium concept for this game is the Ramsey equilibrium.

Private agents actions depend on the prevailing policy. To keep track of that relation, let f denote a generic function that maps a vector (s^t, φ) into the space of the pairs $(\chi(s^t), \zeta(s^t))$. As before, $f(\varphi) = \{[f(s^t,\varphi)]_{s^t\in S^t}\}_{t=0}^{\infty}$. Abusing the notation, $u(f(s^t,\varphi))$ will denote the function u evaluated at the corresponding $(c^T(s^t), c_1^N(s^t), c_2^N(s^t), l(s^t))$ coordinates of $f(s^t, \varphi)$.

Definition 3 *A* Ramsey Equilibrium *is a pair (φ, f) satisfying: (i) for all $\bar{\varphi}$, $f(\bar{\varphi})$ provides solutions for both households' and firms' problems; (ii) φ solves $\max_{\bar{\varphi}} \sum_{t=0}^{\infty}\sum_{s^t\in S^t}\beta^t u(f(s^t,\bar{\varphi}))$ subject to (1) and (2). A triple (φ, χ, ζ) is a* Ramsey outcome *if there exists a f such that (φ, f) is a Ramsey equilibrium and $f(\varphi) = (\chi, \zeta)$.*

Private agents are required to behave optimally for all policies, not only for the equilibrium one. This requirement is a natural consequence of the game being studied. When the government chooses φ, it knows that people and firms will behave optimally, no matter the chosen policy. So, government uses this information when choosing φ. This requirement is equivalent to subgame perfection, as pointed out by Chari and Kehoe [4].

In this world unexpected inflation does not act as a lump sum tax. Therefore, the problem of selecting an optimal policy will have a well defined solution even if the government has some outstanding debt at date zero. See Nicolini [15], specially section 3, for further details.

Recall that $\{g_t^T, g_t^N, \tau_t\}_{t=0}^{\infty}$ is exogenous. Thus, the government problem consists in choosing paths for money supply, domestic debt and external borrowing to maximize people's welfare. One can see this problem as a simplified version of the problem faced by a benevolent central bank that takes the fiscal policy as given.

5.1 Characterization

In a Ramsey equilibrium, the government chooses a policy that will maximize people's welfare. Therefore, it is possible to characterize Ramsey outcomes through a standard maximization problem.

Proposition 2 *Suppose that $(p^N(s^0), \chi)$ solve*

$$\max_{(\bar{p}^N(s^0),\bar{\chi})} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t u(\bar{c}^T(s^t), \bar{c}_1^N(s^t), \bar{c}_2^N(s^t), \bar{l}(s^t))$$

subject to (1) and (9)-(13). If $(p^N(s^0), \chi)$ satisfies (14)-(19), then $(p^N(s^0), \chi)$ is a component of some Ramsey outcome (φ, χ, ζ).

Proof. Suppose that $[p^N(s^0), \chi]$ solves the problem in question. From proposition 1, there exist φ and ζ that supports $[p^N(s^0), \chi]$ as a competitive equilibrium. It will be shown that (φ, χ, ζ) is a Ramsey outcome. For a given $\bar{\varphi}$, define $f(\bar{\varphi})$ as a solution of households' and firms' for this particular $\bar{\varphi}$. Trivially, $f(\varphi) = (\chi, \zeta)$. It remains to show that (φ, f) is a Ramsey equilibrium. The constraints (1) and (9)-(13) are not enough to characterize a competitive equilibrium. Thus, $f(\varphi)$ yields the highest welfare in a set of $\bar{\varphi}$'s that is a superset of all $\bar{\varphi}$'s that can be implemented as a competitive equilibrium. But φ can be implemented as a competitive equilibrium of bounded inflation. Therefore, $f(\varphi)$ yields the highest welfare in the set of $\bar{\varphi}$'s that can be implemented as a competitive equilibrium. Hence, (φ, f) satisfies all conditions of definition 3. ■

Proposition 2 can be read in the following intuitive way: try to maximize people's welfare in a set of allocations that satisfy some, but not all, competitive equilibrium conditions. If it turns out that the solution of the maximization problem satisfies all conditions of a particular type of competitive equilibrium, then this solution is the best competitive equilibrium. This procedure will be used in all incoming examples.

5.2 Examples

This section's goal is to reproduce the stylized facts of section 2. Simple cases will be considered first. The empirical regularities will be reproduced in the last few examples. Details are worked out in the appendix.

Example 1 Consider a standard closed economy with cash and credit goods. The technology is described by $c_{1t} + c_{2t} + g_t = l_t$. The period utility function is $u = \log c_1 + \log c_2 + \log(1 - l)$. Labor income is not taxed, i.e. $\tau_t = 0$, and the initial value of the public debt is zero. The optimal allocation is stationary, in the sense that if $g_t = g_k$ then $(c_{1t}, c_{2t}) = (c_{1k}, c_{2k})$. Government expenditures follow $g_t = \bar{g}$ if $t = \bar{t}$ and $g_t = g$ if $t \neq \bar{t}$, where $0 < g < \bar{g}$ and $\bar{t} \geq 2$. Let (c_1, c_2) and $(\bar{c}_1, \bar{c}_2)$ denote the optimal consumption values associated to those g's. The inflation rate $(p_{t+1} - p_t)/p_t$ is denoted by π_{t+1}. From the household's first order condition, $1 + \pi_{t+1} = \beta c_{2t}/c_{1t+1}$. As shown in the appendix, $c_1 > \bar{c}_1$ and $c_2 > \bar{c}_2$. Hence, $\bar{c}_2/c_1 < c_2/c_1 < c_2/\bar{c}_1$. Thus, inflation starts at some value π_1 and remains constant until date $\bar{t} - 1$. At date $\bar{t}$, as the government consumption grows the inflation jumps. At date $\bar{t} + 1$ people raise their real cash holdings. This additional source of revenue allows the government to reduce the inflation taxation temporarily. At date $\bar{t}+2$ the inflation returns to π_1 and remains constant thereafter.

Example 2 The environment is exactly as in the previous example, except for government consumption. Now, $g_t = \bar{g}$ if t is odd and $g_t = g$ if t is even, with $\bar{g} > g$. Again, $c_1 > \bar{c}_1$ and $c_2 > \bar{c}_2$. So, $\pi_t = \bar{\pi} = \beta c_2/\bar{c}_1 - 1$ for t odd and $\pi_t = \pi = \beta \bar{c}_2/c_1 - 1$ for t even, with $\bar{\pi} > \pi$. Thus, the inflation rate alternates between a high and a low value.

In both examples, the inflation rate is higher in periods of higher public expenditures, although there exist attainable policies that lead to constant inflation. Despite the oscillation in the inflation rate, the government does not balance the budget solely with inflation tax, because the real public debt is not constant (the behavior of the public debt can be checked by a recursive evaluation of the household's lifetime budget constraint).

Lucas and Stokey [12] consider similar models in which the government can choose τ. Although the public debt is used to smooth taxation out over time, τ is not constant. In times of high public expenditures the optimal policies require high tax rates. In the two examples above, inflation is the only taxation instrument available. For this reason, the inflation rate is higher in the states with higher g and lower in the other states.

Next example considers an open economy version of examples 1 and 2. One could expect that inflation and exchange rate devaluation would be high in times of fiscal deficit and low in times of fiscal surplus. However, that will not occur. External borrowing and lending provides the Ramsey planner additional possibilities for smoothing taxation out.

Example 3 A deterministic world is considered. Technology is represented by $c_t^T + c_{1t}^N + c_{2t}^N + x_t + g_t = l_t$. So, $E_t p_t^* = p_t^N = w_t$. As in the previous examples, $\tau_t = 0$. Additionally, assume that $q_t^* = \beta$ and $p_t^* = 1$ and $\bar{B} = \bar{B}_H^* = \bar{B}_G^* = 0$. The period utility function is $u = \log c^T + \log c_1^N + \log c_2^N + \log(1-l)$. From consumer's first order condition, $E_{t+1}/E_t = \beta c_{2t}^N / c_{1t+1}^N$. It is shown in the appendix that c_{2t}^N, c_{1t}^N, c_t^T and l_t are constant. Thus, devaluation rate and inflation are constant. Only x_t changes to offset the variations in g_t. Taxation is fully smoothed out through external borrowing and lending.

In example 3 the shocks on government consumption can be insured through external borrowing and lending at *constant opportunity costs* (because the marginal rate of transformation between tradables and non tradables and the international interest rate are constant). This feature induces the Ramsey planner to completely smooth taxation out. If fiscal shocks cannot be insured at constant opportunity costs, inflation and devaluation rate will not be constant, as in the next example.

The Ramsey policies for the economies considered in the incoming examples 4, 5, and 6 are relatively inexpensive to evaluate with the help of a computer. They illustrate some general properties of the Ramsey equilibrium. So it is worthwhile to study them, although they will not reproduce all stylized facts outlined in section 2.

Example 4 The economy is deterministic. Technology is described by $c_t^T + x_t = \sqrt{l_t^T}$ and $c_{1t}^N + c_{2t}^N + g_t^N = l_t^N$. Again, $q_t^* = \beta$, $p_t^* = 1$ and $\bar{B} = \bar{B}_H^* = \bar{B}_G^* = 0$. The period utility function is as in the previous example. The discount factor satisfies $\beta = 0.98$. The fiscal policy is $(g_t^N, \tau_t) = (0.3, 0.1)$ if t is even and $(g_t^N, \tau_t) = (0.15, 0.2)$ if t is odd. The nominal exchange rate devaluates approximately 11.9% in periods of fiscal deficit (i.e., even dates) and 3.7% in periods of fiscal surplus. Thus, the government does not balance the budget period by period with inflation tax. As the fiscal policy is tightened, the optimal exchange rate policy prescribes a lower, but positive, rate of devaluation for the domestic currency. This is true despite that there exist several alternative policies. For instance, this economy has a competitive equilibrium in which the domestic currency devaluates 8.3% in all periods, another with no devaluation at odd dates and 15.4% devaluation in even dates, and a third with 21.8% in odd periods and no devaluation if t is even.

A decrease in the fiscal deficit has significative effects over the devaluation rate. The consumption of non tradables is negatively correlated to the fiscal deficit, while the GDP is positively correlated to this last variable. Hence, consumption and output are negatively correlated and this is not in accordance with the empirical evidence provided in section 2.

Stochastic economies will be considered now. In examples 5 and 6 $\{\theta_t^N, g_t^N, \tau_t\}_{t=0}^{\infty}$ is a Markov process with state space $\{a, b\}$, where $a = (\theta_a^N, g_a^N, \tau_a)$ and $b = (\theta_b^N, g_b^N, \tau_b)$, and transition probabilities $\mu_{ab} = 0.4$ and $\mu_{ba} = 0.7$. The state space is example specific. At date zero the economy is at state b (the bad state), $p_t^* = 1$, $q_t^* = \beta = 0.98$, $g_t^T = 0$, preferences are as in example 4, and technology is described by $c_1^N(s^t) + c_2^N(s^t) + g_t^N = \theta_t^N l^N(s^t)$ and $c^T(s^t) + x(s^t) = \sqrt{l^T(s^t)}$.

Example 5 In this example, $(\theta_a^N, g_a^N, \tau_a) = (1, 0.2, 0.3)$ and $(\theta_b^N, g_b^N, \tau_b) = (1, 0.4, 0.15)$. The nominal exchange rate devaluates 2.2% at state a, which is the state with fiscal surplus. At state b there is a fiscal deficit and the domestic currency devaluates 17.8%. As in the previous example, the devaluation rate is larger in periods of fiscal deficit and the government does not rely solely on inflation tax to finance temporary fiscal imbalances. Of course, different policies could be pursued. For instance, there exists a competitive equilibrium with a constant devaluation of 9.1% in all states, a second one with no devaluation at state a and 21.1% devaluation at b, and a third one with 19.2% devaluation at state a and no devaluation at state b.

The introduction of stochastic components does not affect the main results found in example 4. Consumption of non tradables and output are negatively correlated. A reduction in the fiscal deficit has strong effects over the optimal devaluation rate.

In both examples 4 and 5 consumption and output are negatively correlated. This contradicts the empirical regularities mentioned in section 2. Therefore, it seems that an economy driven only by fiscal shocks cannot reproduce the stylized facts. That observation motivates the next example.

Example 6 The states are $(\theta_a^N, g_a^N, \tau_a) = (1.25, 0.25, 0.15)$ and $(\theta_b^N, g_b^N, \tau_b) = (0.75, 0.25, 0.15)$. The optimal policy prescribes a nominal devaluation of 19.0% at a and 23.4% at b.

In example 6 both output and consumption of non tradables are negatively correlated with the devaluation rate. However, even an implausible high dispersion for the technological shocks does not induce large oscillations in the devaluation rate.

At this point it is convenient to summarize the findings. The model does explain the successive shifts between periods of low and high devaluation rate. Large oscillations in the devaluation rate with small oscillations in output require fiscal shocks. On the other hand, fiscal deficits and surpluses alone cannot account for the behavior of the output and consumption. These findings will remain true in richer environments. Several non reported experiments corroborate this last assertion.

It is above the scope of this essay to explain why technological and fiscal shocks should be positively correlated. A possible explanation is that the inflation tax is a burden not only for consumers, but also for firms. In a more realistic model, a drop in the devaluation rate could induce supply side effects similar to the technological shocks considered here.

So far, it has been shown that the model can account for the recurrent shift from a regime of low devaluation to one of high devaluation. It remains to be shown that the model can also reproduce the stylized facts of currency crises and peggings. The next two examples will reproduce all these empirical regularities.

Examples 7 and 8 share several feature. The environment is deterministic. The period utility is $u = (c^T)^{\gamma^T}(c_1^N)^{\gamma_1}(c_2^N)^{\gamma_2}(1-l)^{\gamma^l}$. Technology is given by $c_{1t}^N + c_{2t}^N + g_t^N = \theta_t^N (l_t^N)^{\alpha^N}$ and $c_t^T + x_t = \theta_t^T (l_t^T)^{\alpha^T}$. As before, $q_t^* = \beta$, $p_t^* = 1$, and $\bar{B} = \bar{B}_H^* = \bar{B}_G^* = 0$. Time is measured in months.

There exists a large body of quantitative literature on the real effects of an exchange rate pegging. Preferences and technology will be parametrized to make the economy in this paper to resemble some of the models considered in that literature.

Labor income shares $\alpha^T = 0.48$ and $\alpha^N = 0.63$ are borrowed from Rebelo [18] and Rebelo and Végh [19]. They adopted a quarterly value of 0.99 for β. Converting this figure to monthly units, one obtains $\beta = 0.99^{1/3}$, which is the value adopted in the incoming examples. The share parameter γ^l was set equal to 2/3, as in Kydland and Prescott [11]. This last values implies $\gamma^T + \gamma_1 + \gamma_2 = 1/3$. In Rebelo and Végh [19] and Rebelo [18], tradables and non tradables have the same share. Thus, $\gamma^T = \gamma_1 + \gamma_2$. The condition $\gamma_1/\gamma_2 = 2/3$ was imposed arbitrarily on those shares. Solving those three equations, one obtains $\gamma^T = 5/30$, $\gamma_1 = 2/30$ and $\gamma_2 = 3/30$.

The constraint $\gamma_1/\gamma_2 = 2/3$ imply that in a steady state with a low inflation around 40% of the household expenditures with non-tradables will be paid cash. This is a relatively high number. Hence, the adopted parametrization leads to a cash demand larger than one should expect. Given the fiscal policy, a large money demand will lead to smaller inflation rates. So, the adopted parametrization is reducing the model's ability to generate high devaluation rates.

Example 7 This example reproduces the stabilization stylized facts discussed by Kiguel and Liviatan [8] and Végh [21]. These facts are listed at page 3. The economy starts with a loose fiscal policy and low productivity. The domestic currency depreciates 12.7% in nominal terms every month. At the beginning of the fifth year the economy is hit by a positive technological shock and a fiscal deficit reduction, both permanent. That fiscal tightening reduces government expenditures and increases tax revenues. The nominal exchange rate devaluation drops to a monthly value of 1.2%. After the stabilization, consumption and output increase, the current account deteriorates, real wage increases in both sectors, and the domestic currency appreciates in real terms.

Example 8 This example reproduces the currency crisis empirical regularities mentioned at page 3. During the first four years the fiscal policy is tighter and the productivity is high. The nominal exchange rate devaluates only 0.7% every month. At the beginning of the fifth year, a permanent shock reduces the productivity and increase the fiscal deficit. Government expenditures grow and fiscal revenues fall. The new nominal rate of devaluation of the currency is 19.5% monthly. Consumption and output falls and the real exchange rate depreciates.

Several non reported experiments were performed. They suggest that the ability of the model to generate oscillations in the devaluation rate is not sensible to parameter values and functional forms. Hence, this paper successfully explains the first empirical regularity mentioned in section 2.

Given that the model manages to explain the implementation and collapse of pegging policies, the next question is how well the model manages to explain the business cycle regularities (i.e., the second and third sets of facts provided in section 2). The answer to this question requires more care.

In examples 7 and 8, σ was set equal to zero. The qualitative results would remain the same for any value between 0 and 1. For $\sigma \geq 1$, some qualitative features are not reproduced. For instance, the trade balance x becomes cyclical if $\sigma \geq 1$. For further discussion on the connection between preferences and business cycles regularities in an small open economy context, the reader should consult Correa, Neves and Rebelo [5].

It has been shown that the optimal devaluation rate is a function of the economy's state. As the state of the economy changes, the rate of devaluation will oscillate. Hence, pegging episodes and currency crises can be understood as optimal response to exogenous shocks that hit the economy. The empirical regularities associated with pegging and devaluation are reproduced by the competitive equilibrium induced by the Ramsey policy.

6 Conclusion

Governments often choose to pursue exchange rate policies that are later abandoned. To understand the driving forces behind the selection of these policies, this paper studied the problem of choosing an optimal monetary policy with commitment in a context of exogenous fiscal policy. The main finding is that the optimal devaluation rate is correlated in a positive way to the fiscal deficit and in a negative way to technological shocks.

The optimal devaluation policy features have a simple justification. In periods of high fiscal deficit, a benevolent government would like to increase tax rates. If the fiscal policy is exogenous, the only remaining way to raise additional tax revenue is through inflation tax. As the inflation rises, so does the devaluation rate. A negative technological shock will lead to a fall in output. Thus, the ratio between fiscal deficit and output will rise. Again, the government's willingness to raise tax rates explains why the devaluation is higher when there is a bad technology draw.

A currency crisis is often followed by a drop below the trend of consumption and output and a real exchange rate depreciation. When a country pegs the exchange rate, the opposite facts plus a current account deterioration usually take place. Ideally, a model aimed at explaining the implementation and collapse of exchange rate regimes should reproduce these stylized facts. This essay also succeeds in replicating that set of empirical regularities.

There is an intuitive explanation for the link between devaluation and the real side of the economy. A drop in the devaluation rate and an increase in the output will generate an income effect that leads to a consumption boom. The higher demand for tradable goods is partially offset by imports. Thus, the current account deteriorates. Since the higher demand for non tradables cannot be matched by imports, these goods become relatively more expensive. Thus, the real exchange rate appreciates. The opposite occurs when devaluation increases and output falls.

The notion that currency crises trigger recessions is widely accepted. In this essay neither a devaluation causes a slowdown nor a pegging causes a boom. The optimal devaluation rate reacts to technological shocks that hit the economy. If a low productivity shock leads to a recession, to prevent or postpone the devaluation is not an efficient policy.

Most (if not all) of the essays on currency devaluation take an exchange rate pegging as given. However, these papers do not try to explain why the exchange rate was ever pegged. This paper adopts a single framework to explain simultaneously both currency crises and peggings. The model can account for successive shifts between periods of low and high devaluation rate. Related papers account for only one devaluation episode.

This essay has some other contributions. Obstfeld [16] states that it is essential to consider how policies are selected to understand currency crises. Rebelo [18] makes similar statements when discussing monetary stabilization. This paper investigates how the exchange rate policy is chosen.

The essay builds a bridge between two research fields that so far have been seeing as completely apart. Today there is a large and growing body of literature on quantitative macroeconomic theory. Typical examples are the essays of Rebelo [18] and Backus, Kehoe, and Kydland [1]. On the other hand, there exist several studies that rely on reduced form models to explain exchange rate devaluations. Obstfeld [16] and Giavazzi and Pagano [7] are good examples of this investigation avenue. This paper unifies the two approaches.

As shown by Klein and Marion [9], governments often fix the nominal exchange rate. The model does not reproduce this particular fact. Obstfeld [16] and Rebelo and Végh [20] assume that there is a fixed cost of any currency devaluation. The introduction of this feature in the present model is likely to make the optimal policies to prescribe zero devaluation in some states. The introduction of these types of fixed costs is a promising research avenue.

This paper extended the research line started by Lucas and Stokey [12] to an open economy. This allowed the discussion of the optimal exchange rate policy to go beyond the usual discussion of "pegging versus floating". Between those two policies, there are uncountable others. There is no reason to restrain the discussion only to these two extreme options.

7 Appendix

7.1 Households' first order conditions

If $\bar{M}$ is positive, the first order necessary and sufficient conditions for a typical household are

$$\beta^t \mu(s^t) u_T(s^t) = \lambda(s^t) E(s^t) p_t^* \ ; \tag{20}$$

$$\beta^t \mu(s^t) u_1(s^t) = [\lambda(s^t) + \xi(s^t)] p^N(s^t) \ ; \tag{21}$$

$$\beta^t \mu(s^t) u_2(s^t) = \lambda(s^t) p^N(s^t) \ ; \tag{22}$$

$$-\beta^t \mu(s^t) u_l(s^t) = \lambda(s^t)(1-\tau_t) w(s^t) \ ; \tag{23}$$

$$\lambda(s^t) = \sum_{s_{t+1} \in S} [\lambda(s^t, s_{t+1}) + \xi(s^t, s_{t+1})] \ ; \tag{24}$$

$$\lambda(s^t) q(s^t) = \sum_{s_{t+1} \in S} \lambda(s^t, s_{t+1}) \ ; \tag{25}$$

$$\lambda(s^t) E(s^t) q_t^* = \sum_{s_{t+1} \in S} \lambda(s^t, s_{t+1}) E(s^t, s_{t+1}) \ ; \tag{26}$$

$$M(s^{t-1}) \geq p^N(s^t) c_1^N(s^t) \ \& \ \xi(s^t)[M(s^{t-1}) - p^N(s^t) c_1^N(s^t)] = 0 \ ; \tag{27}$$

$$\begin{gathered} E(s^t) p_t^* c^T(s^t) + p^N(s^t)[c_1^N(s^t) + c_2^N(s^t)] + q(s^t) B(s^t) + E(s^t) q_t^* B_H^*(s^t) + \\ M(s^t) = (1-\tau_t) w(s^t) l(s^t) + B(s^{t-1}) + E(s^t) B_H^*(s^{t-1}) + M(s^{t-1}) \ ; \end{gathered} \tag{28}$$

$$\lim_{t \to \infty} \sum_{s^t \in S_k^t} \lambda(s^t) M(s^t) = \lim_{t \to \infty} \sum_{s^t \in S_k^t} \lambda(s^t) q(s^t) B(s^t) = \lim_{t \to \infty} \sum_{s^t \in S_k^t} \lambda(s^t) E(s^t) q_t^* B_H^*(s^t) = 0 \ ; \tag{29}$$

$$c^T(s^t), c_1^N(s^t), c_2^N(s^t), l(s^t), M(s^t), \lambda(s^t), \xi(s^t) \geq 0 \ , \ l(s^t) \leq 1 \ ; \tag{30}$$

$$\left\|c^T\right\|_\infty, \left\|c_1^N\right\|_\infty, \left\|c_2^N\right\|_\infty, \|l\|_\infty, \left\|M/p^N\right\|_\infty, \left\|B/p^N\right\|_\infty, \|B_H^*/p^*\|_\infty < \infty \ ; \tag{31}$$

where $\lambda(s^t)$ and $\xi(s^t)$ are Lagrange multipliers for, respectively, budget and cash-in-advance constraints.

7.2 Proof of proposition 1

For the "if" part, suppose that (φ, χ, ζ) is a competitive equilibrium of bounded inflation. It is needed to show that (1) and (9)-(19) hold. Constraint (1) is trivially satisfied.

It will now be shown that (9) holds. Multiplying (28) by $\lambda(s^t)$ and using (20)-(27) one obtains

$$\begin{gathered} \beta^t \mu(s^t) W(s^t) + \sum_{s_{t+1}} [\lambda(s^t, s_{t+1}) + \xi(s^t, s_{t+1})] M(s^t) - [\lambda(s^t) + \xi(s^t)] M(s^{t-1}) + \\ \lambda(s^t) q(s^t) B(s^t) - \lambda(s^t) B(s^{t-1}) + \lambda(s^t) E(s^t) q_t^* B_H^*(s^t) - \lambda(s^t) E(s^t) B_H^*(s^{t-1}) = 0 \ . \end{gathered} \tag{32}$$

Adding up over s^t and then from date 0 to some date k and using (25) and (26) to cancel the identical terms out one gets

$$\begin{gathered} u_T(s^0) c^T(s^0) + u_2(s^0) c_2^N(s^0) + u_l(s^0) l(s^0) + \sum_{t=1}^{k} \sum_{s^t} \beta^t \mu(s^t) W(s^t) + \\ [u_1(s^0) - \xi(s^0) p^N(s^0)] c_1^N(s^0) + \sum_{s^k} \sum_{s_{k+1}} [\lambda(s^k, s_{k+1}) + \xi(s^k, s_{k+1})] M(s^k) + \end{gathered}$$

$$\sum_{s^k} \lambda(s^k)[q(s^k)B(s^k) + E(s^k)q_k^* B_H^*(s^k)] = \lambda(s^0)[\bar{M} + \bar{B} + E(s^0)\bar{B}_H^*] \ .$$

But $u_1(s^0) - \xi(s^0)p^N(s^0) = \lambda(s^0)p^N(s^0)$. So, the last equality combined to (24) yields

$$u_T(s^0)c^T(s^0) + u_2(s^0)c_2^N(s^0) + u_l(s^0)l(s^0) + \sum_{t=1}^{k}\sum_{s^t} \beta^t \mu(s^t)W(s^t) =$$

$$\lambda(s^0)[\bar{B} + \bar{M} - p^N(s^0)c_1^N(s^0) + E(s^0)\bar{B}_H^*] - \sum_{s^k} \lambda(s^k)[M(s^k) + q(s^k)B(s^k) + E(s^k)q_k^* B_H^*(s^k)] \ . \quad (33)$$

From (20) and (22), $\lambda(s^0)E(s^0) = u_T(s^0)/p_0^T$ and $\lambda(s^0) = u_2(s^0)/p^N(s^0)$. Plugging those two expressions into (33), making $k \to \infty$, using (29) and adding $u_1(s^0)c_1^N(s^0)$ one obtains (9).

Equation (7) has to hold in a competitive equilibrium. Multiplying it by $\lambda(s^t)E(s^t)$, adding up over s^t and from date 0 to some date k and applying (26) to cancel the identical terms out one obtains

$$\lambda(s^0)E(s^0)[\bar{B}_H^* + \bar{B}_G^*] = -\sum_{t=0}^{k}\sum_{s^t} \lambda(s^t)E(s^t)p_t^* x(s^t) + \sum_{s^k} \lambda(s^k)E(s^k)q_k^*[B_H^*(s^k) + B_G^*(s^k)] \ . \quad (34)$$

For a while, assume that

$$\lim_{t\to\infty} \sum_{s^t \in S_k^t} \lambda(s^t)E(s^t)q_t^* B_G^*(s^t) = 0 \ . \quad (35)$$

So, making $k \to \infty$, applying the transversality conditions in (35) and (29), and using (20) one obtains (10). To show that (35) holds, let $\bar{q}^* = \sup Q^*$ and $\bar{p}^* = \sup P^*$. Then,

$$0 \le \left| \sum_{s^t \in S_k^t} \lambda(s^t)E(s^t)q_t^* B_H^*(s^t) \right| \le \bar{q}^* \bar{p}^* \left\| \frac{B_G^*}{p^*} \right\|_\infty \sum_{s^t \in S_k^t} \lambda(s^t)E(s^t) \ .$$

Since $\sum_{s^t \in S_k^t} \lambda(s^t)E(s^t) \le \sum_{s^t \in S^t} \lambda(s^t)E(s^t)$

$$0 \le \left| \sum_{s^t \in S_k^t} \lambda(s^t)E(s^t)q_t^* B_H^*(s^t) \right| \le \bar{q}^* \left\| \frac{B_G^*}{p^*} \right\|_\infty \bar{p}^* \sum_{s^t \in S^t} \lambda(s^t)E(s^t) \ .$$

It is now enough to show that $\sum_{s^t \in S^t} \lambda(s^t)E(s^t) \to 0$ as $t \to \infty$. Add both sides of (26) over s^t. This yields

$$\bar{q}^* \sum_{s^t \in S^t} \lambda(s^t)E(s^t) \ge \sum_{s^{t+1} \in S^{t+1}} \lambda(s^{t+1})E(s^{t+1}) \ .$$

Since $\bar{q}^* \in (0,1)$, (35) is established.

It will now be shown that (11), (12) and (13) hold. Fix s^t. Divide both sides of (20) by p_t^*. Then, forward it by one period, add over s_{t+1} and combine the resulting equation to (26) and (20) to obtain (11). For (12), divide (20) by (22) and combine the resulting equation to item (ii) of definition 1. To obtain (13) divide (23) by (22). Then, use item (ii) of definition 1. This procedures generates (13).

Constraint (14) is obviously satisfied. Concerning (15), divide (21) by (22) to obtain

$$\frac{u_1(s^t)}{u_2(s^t)} = 1 + \frac{\xi(s^t)}{\lambda(s^t)} \ge 1 \ .$$

For (16), note that

$$\sum_{s^t} \lambda(s^t)M(s^t) = \sum_{s^t}\sum_{s_{t+1}} M(s^t)[\lambda(s^t,s_{t+1}) + \xi(s^t,s_{t+1})] =$$

$$\beta^{t+1}\sum_{s^t}\sum_{s_{t+1}} \mu(s^t,s_{t+1})M(s^t)\frac{u_1(s^t,s_{t+1})}{p^N(s^t,s_{t+1})} \geq$$

$$\beta^{t+1}\sum_{s^t}\sum_{s_{t+1}} \mu(s^t,s_{t+1})\left[p^N(s^t,s_{t+1})c_1^N(s^t,s_{t+1})\frac{u_1(s^t,s_{t+1})}{p^N(s^t,s_{t+1})}\right] =$$

$$\beta^{t+1}\sum_{s^{t+1}} \mu(s^{t+1})u_1(s^{t+1})c_1^N(s^{t+1}) \geq 0 \ . \tag{36}$$

Now make $t \to \infty$ and apply (29) to obtain (16).

To obtain (17), proceed exactly as done to obtain (34). However, instead of adding from date zero to k, add from some generic date j to k. This procedure yields

$$\lambda(s^j)E(s^j)[B_H^*(s^j) + B_G^*(s^j)] = -\sum_{t=j}^{k}\sum_{s^t\in S_j^t} \lambda(s^t)E(s^t)p_t^* x(s^t) +$$

$$\sum_{s^k\in S_j^k} \lambda(s^k)E(s^k)q_k^*[B_H^*(s^k) + B_G^*(s^k)] \ . \tag{37}$$

From (29) and (35), the second term in the right hand side goes to zero as $k \to \infty$. Hence, combine (20) and (37) to obtain

$$\sup_j \sup_{s^j\in S^j} \left| \frac{1}{u_T(s^j)} \sum_{t=j}^{\infty}\sum_{s^t\in S^t} \beta^{t-j}\mu(s^t|s^j)u_T(s^t)x(s^t)\right| \leq \frac{\sup P^*}{\inf P^*}\left(\left\|\frac{B_H^*}{p^*}\right\|_\infty + \left\|\frac{B_G^*}{p^*}\right\|_\infty\right) < \infty \ .$$

Regarding (18), add up (32) over s^t and then from date j to date k. With some manipulation, the result is

$$\frac{1}{u_2(s^j)}\left|\sum_{t=j}^{\infty}\sum_{s^t\in S^t} \beta^{t-j}\mu(s^t|s^j)W(s^t) - u_1(s^j)c_1^N(s^j)\right| \leq$$

$$\frac{p^N(s^{j-1})}{p^N(s^j)}\left(\left|\frac{M(s^{j-1})}{p^N(s^{j-1})}\right| + \left|\frac{B(s^{j-1})}{p^N(s^{j-1})}\right|\right) + \left|c_1^N(s^j)\right| + \frac{E(s^j)p_j^*}{p^N(s^j)}\frac{p_{j-1}^*}{p_j^*}\left|\frac{B_H^*(s^{j-1})}{p_{j-1}^*}\right| \leq$$

$$\frac{1}{\varepsilon}\left(\left\|\frac{M}{p^N}\right\|_\infty + \left\|\frac{B}{p^N}\right\|_\infty\right) + \sup\Theta^N + \frac{E(s^j)p_j^*}{p^N(s^j)}\frac{\sup P^*}{\inf P^*}\left\|\frac{B_H^*}{p^*}\right\|_\infty \ . \tag{38}$$

On the other hand,

$$\frac{E(s^j)p_j^*}{p^N(s^j)} = \frac{\alpha^N\theta_j^N[l^T(s^j)]^{1-\alpha^T}}{\alpha^T\theta_j^T[l^N(s^j)]^{1-\alpha^N}} \leq \frac{\alpha^N\sup\Theta^N}{\alpha^T\inf\Theta^T[l^N(s^j)]^{1-\alpha^N}} \ .$$

But $\inf G^N > 0$. So, $l^N(s^t)$ is bounded away from zero. This implies that the right hand side of (38) is bounded by some real number. As a consequence, (18) holds.

The "if" part of the proof will be concluded by showing that (19) is satisfied. Assume that there exists a uniform $\bar{\varepsilon} > 0$ such that

$$\bar{\varepsilon} \leq \frac{\beta}{\mu(s^t)u_2(s^t)}\sum_{\hat{s}_{t+1}} \mu(s^t,\hat{s}_{t+1})u_1(s^t,\hat{s}_{t+1}) \leq \frac{1}{\bar{\varepsilon}} \ , \tag{39}$$

$$\bar{\varepsilon} \leq \frac{c_1^N(s^t, \hat{s}_{t+1})}{c_1^N(s^t, s_{t+1})} \leq \frac{1}{\bar{\varepsilon}} \ . \tag{40}$$

Fix a history s^{t+1}. Pick $\bar{s}_{t+1}$ and $\tilde{s}_{t+1}$ so that $\frac{c_1^N(s^t,\bar{s}_{t+1})}{c_1^N(s^t,\tilde{s}_{t+1})}$ is the smallest value of $\frac{c_1^N(s^t,\hat{s}_{t+1})}{c_1^N(s^t,s_{t+1})}$ over all possible pairs $(s_{t+1}, \hat{s}_{t+1})$. Therefore,

$$\bar{\varepsilon}^2 \leq \frac{\beta}{\mu(s^t)u_2(s^t)} \sum_{\hat{s}_{t+1}} \mu(s^t, \hat{s}_{t+1}) u_1(s^t, \hat{s}_{t+1}) \frac{c_1^N(s^t, \bar{s}_{t+1})}{c_1^N(s^t, \tilde{s}_{t+1})} \Rightarrow$$

$$\bar{\varepsilon}^2 \leq \frac{\beta}{\mu(s^t)u_2(s^t)} \sum_{\hat{s}_{t+1}} \mu(s^t, \hat{s}_{t+1}) u_1(s^t, \hat{s}_{t+1}) \frac{c_1^N(s^t, \hat{s}_{t+1})}{c_1^N(s^t, s_{t+1})} \ .$$

This establishes the first inequality in (19). Similar reasoning yields the second one. It only remains to show that there exists a $\bar{\varepsilon}$ as in (39) and (40). For the left inequality in (39), note that

$$\varepsilon \leq \frac{p^N(s^t, s_{t+1})}{p^N(s^t)} \leq \frac{1}{\varepsilon} \Rightarrow \varepsilon^2 \leq \frac{p^N(s^t, \bar{s}_{t+1})}{p^N(s^t, \hat{s}_{t+1})} \leq \frac{1}{\varepsilon^2} \ .$$

Equations (21), (22) and (24) together imply

$$\varepsilon \leq \frac{p^N(s^t, s_{t+1})}{p^N(s^t)} = \frac{\beta}{\mu(s^t)u_2(s^t)} \sum_{\hat{s}_{t+1}} \mu(s^t, \hat{s}_{t+1}) u_1(s^t, \hat{s}_{t+1}) \frac{p^N(s^t, s_{t+1})}{p^N(s^t, \hat{s}_{t+1})} \leq \frac{1}{\varepsilon} \ .$$

Combine the last two expressions to obtain

$$\varepsilon \leq \frac{\beta}{\mu(s^t)u_2(s^t)} \sum_{\hat{s}_{t+1}} \mu(s^t, \hat{s}_{t+1}) u_1(s^t, \hat{s}_{t+1}) \frac{1}{\varepsilon^2} \Rightarrow \varepsilon^3 \leq \frac{\beta}{\mu(s^t)u_2(s^t)} \sum_{\hat{s}_{t+1}} \mu(s^t, \hat{s}_{t+1}) u_1(s^t, \hat{s}_{t+1}) \ .$$

Similar reasoning shows that the right inequality in (39) holds. To establish (40), it will be shown that if that condition fails then the optimality by households will be violated. Without loss of generality, assume that right inequality in (40) fails. Hence, by taking a subsequence $\{t_k\}_{k=0}^{\infty}$ if necessary, for each t one can find histories (s^t, s_{t+1}) and $(s^t, \hat{s}_{t+1})$ such that $\frac{c_1^N(s^t,\hat{s}_{t+1})}{c_1^N(s^t,s_{t+1})} \to \infty$. Since c_1^N is bounded above, $c_1^N(s^t, s_{t+1}) \to 0$, from which follows that $u_1(s^t, s_{t+1}) \to \infty$. On the other hand, the ratio $\frac{p^N(s^t,\hat{s}_{t+1})}{p^N(s^t,s_{t+1})}$ is bounded away from zero. Thus,

$$\frac{p^N(s^t, \hat{s}_{t+1}) c_1^N(s^t, \hat{s}_{t+1})}{p^N(s^t, s_{t+1}) c_1^N(s^t, s_{t+1})} \to \infty \Rightarrow \frac{M(s^t)}{p^N(s^t, s_{t+1}) c_1^N(s^t, s_{t+1})} \to \infty \ .$$

So, for t sufficiently large, $M(s^t) > p^N(s^t, s_{t+1}) c_1^N(s^t, s_{t+1})$. But not to spend cash holdings fully can not be an optimal choice when $u_1(s^t, s_{t+1}) \to \infty$.

For the "only if" part of the proposition, take an initial price $p^N(s^0) > 0$ and an object χ satisfying (1) and (9)-(19). It must be shown that there exist a pair (φ, ζ) that satisfy all conditions of a competitive equilibrium of bounded inflation.

Recall that $p^N(s^0)$ is given. Thus, it is possible to define $p^N(s^{t+1})$ recursively. Set those prices according to

$$p^N(s^t, s_{t+1}) = \frac{\beta p^N(s^t)}{\mu(s^t)u_2(s^t)} \sum_{\hat{s}_{t+1}} \mu(s^t, \hat{s}_{t+1}) u_1(s^t, \hat{s}_{t+1}) \frac{c_1^N(s^t, \hat{s}_{t+1})}{c_1^N(s^t, s_{t+1})} \ . \tag{41}$$

Set $\lambda(s^t)$ as in (22), $\xi(s^t)$ as in (21), $E(s^t)$ as in (20), $q(s^t)$ as in (25) and $w(s^t)$ as in (23).

From (41), $p^N(s^t, s_{t+1})c_1^N(s^t, s_{t+1}) = p^N(s^t, \hat{s}_{t+1})c_1^N(s^t, \hat{s}_{t+1})$. Thus, one can define cash holdings as $M(s^t) = p^N(s^t, s_{t+1})c_1^N(s^t, s_{t+1})$. Let $B_H^*(s^t) = 0$. Define $B(s^1)$ to balance household's budget constraint at s^1. The entire array $\{[B(s^t)]_{s^t \in S^t}\}_{t=0}^{\infty}$ is constructed in this recursive way, while $\{[B_G^*(s^t)]_{s^t \in S^t}\}_{t=0}^{\infty}$ is defined recursively to balance the budget constraint of the government.

It remains to show that the proposed (φ, χ, ζ) is a competitive equilibrium of bounded inflation. Combining (41) and (19) it is easy to check that inflation is bounded. Except for the condition $\|B_G^*/p^*\|_\infty < \infty$ (which will be established at the end of the proof), item (iii) of definition 1 is clearly satisfied. For item (i) it is enough to prove that (20)-(30) are satisfied. The variables were defined so that (20)-(23) hold. Concerning (24), from (41) one obtains

$$\frac{\beta^t \mu(s^t) u_2(s^t)}{p^N(s^t)} = \beta^{t+1} \sum_{\hat{s}_{t+1}} \frac{\mu(s^t, \hat{s}_{t+1}) u_1(s^t, \hat{s}_{t+1}) c_1^N(s^t, \hat{s}_{t+1})}{p^N(s^t, s_{t+1}) c_1^N(s^t, s_{t+1})} =$$

$$\beta^{t+1} \sum_{\hat{s}_{t+1}} \frac{\mu(s^t, \hat{s}_{t+1}) u_1(s^t, \hat{s}_{t+1}) c_1^N(s^t, \hat{s}_{t+1})}{p^N(s^t, \hat{s}_{t+1}) c_1^N(s^t, \hat{s}_{t+1})} \Rightarrow \frac{\beta^t \mu(s^t) u_2(s^t)}{p^N(s^t)} = \beta^{t+1} \sum_{\hat{s}_{t+1}} \frac{\mu(s^t, \hat{s}_{t+1}) u_1(s^t, \hat{s}_{t+1})}{p^N(s^t, \hat{s}_{t+1})} .$$

The last equality combined to (22) generates (24).

Debt prices $q(s^t)$ were defined so that (25) holds. Combining (20) and (11) one obtains (26). Concerning (27), (14) implies that it holds in state s^0 and cash holdings were defined so that $M(s^{t-1}) = p^N(s^t)c_1^N(s^t)$ for $t \geq 1$. The definition of $B(s^t)$ guarantees that (28) holds.

Recall that $B_H^*(s^t) = 0$. Thus, the last limit in (29) holds. Concerning the first limit, variables were constructed so that (36) is satisfied, with the first inequality holding as equality. So, (16) implies that $\sum_{s^t} \lambda(s^t) M(s^t) \to 0$ as $t \to \infty$. For the second limit, observe that (33) can be derived exactly as before. Plus, (9) ensures that $\sum_{t=1}^{\infty} \sum_{s^t} \beta^t \mu(s^t) W(s^t)$ converges in $\mathbb{R}$. So, making $k \to \infty$ in (33) and using the fact that the other two transversality conditions in (29) hold one concludes that $\lim_{k \to \infty} \sum_{s^k} \lambda(s^k) q(s^k) B(s^k) = 0$.

With the exception of $\xi(s^t) \geq 0$ and $\|B/p^N\|_\infty < \infty$, all inequalities in (30) and (31) are trivially true. To show that former holds, divide (21) by (22) and use (15). With respect to the latter, the same procedure used to obtain (38) yields

$$\frac{p^N(s^j)}{p^N(s^{j-1})} \frac{1}{u_2(s^j)} \left| \sum_{t=j}^{\infty} \sum_{s^t \in S^t} \beta^{t-j} \mu(s^t | s^j) W(s^t) - u_1(s^j) c_1^N(s^j) \right| = \left| \frac{B(s^{j-1})}{p^N(s^{j-1})} \right| .$$

Constraints (18) and (19) imply that the right hand side is bounded. Thus, $\|B/p^N\|_\infty < \infty$.

The focus is now on item (ii) of definition 1. Combine (22), (23) and (13) to obtain the first equality. Divide (20) by (22), combine the resulting equality to (12) and use the equality $w(s^t)/p^N(s^t) = \alpha^N \theta_t^N [l^N(s_t)]^{\alpha^N - 1}$ to obtain the second equality.

To finish the proof it only remains to show that $\|B_G^*/p^*\|_\infty < \infty$. Combine (20) and (37) to get

$$\frac{p_j^*}{p_{j-1}^*} \left| \frac{1}{u_T(s^j)} \sum_{t=j}^{\infty} \sum_{s^t \in S^t} \beta^{t-j} \mu(s^t | s^j) u_T(s^t) x(s^t) \right| = \left| \frac{B_G^*(s^{j-1})}{p_{j-1}^*} \right| .$$

Since P^* is a finite set, an appeal to (17) concludes. ∎

7.3 Model's National Accounts

Let e, c and y denote, respectively, real exchange rate, consumption and output. These variables are quantified according to $e = Ep^*/p^N = u_T/u_2$ (the second equality comes from households' first order conditions), $c = c_1^N + c_2^N + ec^T$ and $y = y^N + ey^T = \theta^N (l^N)^{\alpha^N} + e\theta^T (l^T)^{\alpha^T}$. The current account is identical to the trade balance x. Sectorial real wages can be evaluated by marginal productivities. The term fiscal deficit refers to primary deficit, which is the difference between expenditures $g^N + eg^T$ and fiscal revenue $\tau\theta^N(l^N)^{\alpha^N-1}l + (1-\alpha^N)\theta^N(l^N)^{\alpha^N} + e(1-\alpha^T)\theta^T(l^T)^{\alpha^T}$. In this sum, the first term is the labor income tax revenue and the other two correspond to profits.

7.4 Examples' Solutions

All Ramsey examples use proposition 2. In each exercise, the lifetime utility (4) is maximized subject to the constraints (1) and (9)-(13). Since the solution also satisfies (14)-(19), that solution is a Ramsey allocation. A proper choice for $p^N(s^0)$ will ensure that (14) holds. For examples (1)-(3) it will be proved that (15) is satisfied. For the other examples, a numerical verification shows that (15) holds. Concerning constraints(16)-(19), they are surely satisfied because in each example the allocation takes only finitely many values.

It is a well known fact in the Ramsey policies the government uses distorting taxation only after using all available lump-sum revenues. Particularly, if the public expenditures are high enough the government will always be willing to raise all possible lump-sum revenue at date zero. This implies that the date zero cash-in-advance constraint will hold as equality. Otherwise, the money holdings left over would consist on wealth not taxed away through inflation in a lump-sum fashion.

The above property will be used in all examples. Assuming that the date zero cash-in-advance hold as equality, the right hand side of (9) can be simplified. Since in all solutions the government will use the inflation tax, the assumption in question is justified.

Examples 1, 2 and 3 were solved analytically and their solutions will be discussed in the details. Examples 4 to 8 were solved numerically. The steps of the solution algorithm will also be detailed.

Examples 1 and 2. In this environment, consumers face cash-in-advance $p_t c_{1t} \leq M_{t-1}$ and budget $p_t(c_{1t}+c_{2t}) + M_t + q_t B_t \leq p_t l_t + M_{t-1} + B_{t-1}$ constraints. The government budget constraint is $p_t g_t + M_{t-1} + B_{t-1} = M_t + q_t B_t$. A convenient superset of the competitive equilibrium allocations in which the date zero cash-in-advance constraint holds as equality is characterized by $c_{1t}+c_{2t}+g_t = l_t$, $\sum_{t=0}^{\infty}\beta^t \frac{l_t}{1-l_t} = \frac{1+\beta}{1-\beta}$ and $c_{2t} = 1 - l_t$. Plugging the last equation into the other two one obtains $c_{1t} + 2c_{2t} + g_t = 1$ and $\sum_{t=0}^{\infty}\frac{\beta^t}{c_{2t}} = \frac{2+\beta}{1-\beta}$. The period utility function becomes $u = \log c_1 + 2\log c_2$. Thus, the Ramsey problem is to maximize $\sum_{t=0}^{\infty}\beta^t(\log c_{1t} + 2\log c_{2t})$ subject those two constraints. Let Λ be the Lagrange multiplier for the second one. The value of Λ depends on the entire sequence $\{g_t\}_{t=0}^{\infty}$. Given Λ, the date t allocation is fully characterized by the first constraint and

$$\frac{1}{c_{2t}} + \frac{\Lambda}{2(c_{2t})^2} = \frac{1}{c_{1t}} \ . \tag{42}$$

Thus, the solution is stationary, in the sense that if $g_t = g_k$ then $(c_{1t}, c_{2t}) = (c_{1k}, c_{2k})$. It will now be shown that $c_{1t} \leq c_{2t}$, which is the equivalent of constraint (15). Note that

$$\Lambda = 2(c_{2t})^2\left(\frac{c_{2t}-c_{1t}}{c_{1t}c_{2t}}\right) \ .$$

Thus, if $c_{1t} > c_{2t}$ for some t, then the same holds for all t. So, it is enough to show that $c_{1t} > c_{2t}$ for all t leads to a contradiction. Combining that inequality to $c_{1t} + 2c_{2t} + g_t = 1$ one obtains

$$3c_{2t} < 1 - g_t \Rightarrow \frac{\beta^t}{c_{2t}} > \frac{3\beta^t}{1-g_t} > 3\beta^t \Rightarrow \sum_{t=0}^{\infty}\frac{\beta^t}{c_{2t}} > 3\sum_{t=0}^{\infty}\beta^t > \frac{2+\beta}{1-\beta} = \sum_{t=0}^{\infty}\frac{\beta^t}{c_{2t}} \ ,$$

which is a contradiction. This also implies that $\Lambda > 0$. It remains to show that $c_1 > \bar{c}_1$ and $c_2 > \bar{c}_2$. The constraint $c_{1t} + 2c_{2t} + g_t = 1$ implies that at least one of those inequalities hold. From (42),

$$\frac{c_2 - \bar{c}_2}{c_2\bar{c}_2}\left[1 + \frac{\Lambda}{2}\left(\frac{1}{c_2} + \frac{1}{\bar{c}_2}\right)\right] = \frac{c_1 - \bar{c}_1}{c_1\bar{c}_1} \ .$$

Since $\Lambda > 0$, the term inside brackets is positive. Thus, $c_2 - \bar{c}_2$ and $c_1 - \bar{c}_1$ have the same signal, from which follows that both inequalities $c_1 > \bar{c}_1$ and $c_2 > \bar{c}_2$ hold. Note that l_t is not constant. Hence, a recursive evaluation of $\sum_{t=0}^{\infty} \beta^t \frac{l_t}{1-l_t}$ shows that people's real assets are not constant. Thus, the optimal policy prescribe public debt issuing.

It was shown, by means of algebraic manipulations, that the solutions of examples 1 and 2 are both stationary. All examples discussed in this paper have the same property. There is an intuitive reason for this. The maximization problem whose solution provides the optimal allocations has no endogenous state variable. So, the solutions turn out to be stationary.

Example 3. The relevant constraints are $c_t^T + c_{1t}^N + c_{2t}^N + x_t + g_t = l_t$, $\sum_{t=0}^{\infty} \beta^t \frac{l_t}{1-l_t} = \frac{1-\beta}{3+\beta}$, $\sum_{t=0}^{\infty} \beta^t x_t / c_t^T = 0$, $c_{t+1}^T = c_t^T$, $c_{2t}^N = c_t^T$, $1 - l_t = c_{2t}^N$. Clearly, c_t^T and c_{2t}^N are constant. Hence, $c_{2t}^N = c_t^T = 1 - l_t = (3+\beta)^{-1}$. The Ramsey problem is to maximize $\sum_{t=0}^{\infty} \beta^t \log c_{1t}^N$ subject to $c_{1t}^N + x_t + g_t = \frac{\beta}{3+\beta}$ and $\sum_{t=0}^{\infty} \beta^t x_t = 0$. Let Γ be a Lagrange multiplier for $\sum_{t=0}^{\infty} \beta^t x_t = 0$. The optimal allocation satisfies $c_{1t}^N = \Gamma^{-1}$. It remains to show that $c_{1t}^N \leq (3+\beta)^{-1}$. Assume that $c_{1t}^N > (3+\beta)^{-1}$. So, $x_t < -g_t - \frac{1-\beta}{3+\beta} < 0$. But this contradicts $\sum_{t=0}^{\infty} \beta^t x_t = 0$.

As previously mentioned, examples 4 to 8 were solved numerically (as well as the concurrent examples of competitive equilibrium). The well-know Newton's method (see Burden and Faires [2]) was used. Solutions were computed with a maximum error of 10^{-9}. As a consequence, the allocations were evaluated with several decimals. However, the space constraint required them to be present with four decimal places only.

Examples 4, 5, 6. These exercises share several common features. Therefore, it is possible to address all of them in a general framework (the deterministic economy is a particular case of the stochastic framework). The relevant constraints are $c^T(s^t) + x(s^t) = \theta_t^T \sqrt{l^T(s^t)}$, $c_1^N(s^t) + c_2^N(s^t) + g_t^N = \theta_t^N l^N(s^t)$, $l^T(s^t) + l^N(s^t) = l(s^t)$, $\sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \mu(s^t) \frac{l(s^t)}{1-l(s^t)} = \frac{2+\beta}{1-\beta}$, $\sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \mu(s^t) \frac{x(s^t)}{c^T(s^t)} = 0$, $\theta_t^T c_2^N(s^t) = 2\theta_t^N c^T(s^t) \sqrt{l^T(s^t)}$, $(1-\tau_t)\theta_t^N [1 - l(s^t)] = c_2^N(s^t)$, plus

$$\frac{1}{c^T(s^{t-1}, a)} = \frac{1-\mu_{ab}}{c^T(s^t, a)} + \frac{\mu_{ab}}{c^T(s^t, b)} \text{ and } \frac{1}{c^T(s^{t-1}, b)} = \frac{\mu_{ba}}{c^T(s^t, a)} + \frac{1-\mu_{ba}}{c^T(s^t, b)} \ . \tag{43}$$

In the deterministic case, equations (43) became a single one and c_t^T is constant in any competitive equilibrium. In the stochastic case, $c^T(s^t)$ does not need to be constant. However, a analysis of the first order conditions shows that the Ramsey problem admits stationary solutions. In this case, the optimal allocation will satisfy $c^T(s^t) = c^T$. Therefore, there is no loss of generality in replacing (43) by $c^T(s^t) = c^T$ in the formulation of the maximization problem. Thus, the Ramsey problem is to maximize $\sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \mu(s^t) \left\{ \log c^T + \log c_1^N(s^t) + \log c_2^N(s^t) + \log[1 - l(s^t)] \right\}$ subject to the constraints listed above.

After some manipulation, the first order conditions can be expressed as a 17×17 non linear system of equations. Example 4 is a particular case in which $\mu_{ab} = \mu_{ba} = 1$. The Ramsey and the respective competitive equilibrium (CE) allocations for each example are displayed next.

TABLE 1

EXAMPLE 4 ALLOCATIONS

	t	c_t^T	c_{1t}^N	c_{2t}^N	l_t	l_t^N	l_t^T	x_t
Ramsey	even	0.3280	0.1773	0.2025	0.7750	0.6798	0.0952	−0.0194
	odd	0.3280	0.2155	0.2282	0.7147	0.5937	0.1210	0.0198
1st CE	even	0.3283	0.1819	0.2010	0.7767	0.6829	0.0937	−0.0221
	odd	0.3283	0.2085	0.2304	0.7120	0.5889	0.1231	0.0226
2nd CE	even	0.3278	0.1731	0.2038	0.7735	0.6769	0.0966	−0.0170
	odd	0.3278	0.2218	0.2263	0.7171	0.5980	0.1191	0.0173
3rd CE	even	0.3291	0.1934	0.1974	0.7807	0.6907	0.0899	−0.0292
	odd	0.3291	0.1900	0.2361	0.7048	0.5761	0.1287	0.0298

TABLE 2

EXAMPLE 5 ALLOCATIONS

	s_t	$c^T(s^t)$	$c_1^N(s^t)$	$c_2^N(s^t)$	$l(s^t)$	$l^N(s^t)$	$l^T(s^t)$	$x(s^t)$
Ramsey	a	0.3104	0.1958	0.2042	0.7083	0.6000	0.1082	0.0186
	b	0.3104	0.1443	0.1735	0.7959	0.7178	0.0781	−0.0309
1st CE	a	0.3110	0.1860	0.2071	0.7040	0.5932	0.1109	0.0220
	b	0.3110	0.1553	0.1707	0.7992	0.7239	0.0753	−0.0366
2nd CE	a	0.3102	0.1991	0.2032	0.7097	0.6024	0.1073	0.0174
	b	0.3102	0.1411	0.1745	0.7947	0.7156	0.0791	−0.0289
3rd CE	a	0.3119	0.1735	0.2109	0.6987	0.5844	0.1143	0.0262
	b	0.3119	0.1640	0.1673	0.8032	0.7312	0.0719	−0.0437

TABLE 3

EXAMPLE 6 RAMSEY ALLOCATION

s_t	$c^T(s^t)$	$c_1^N(s^t)$	$c_2^N(s^t)$	$l(s^t)$	$l^N(s^t)$	$l^T(s^t)$	$x(s^t)$
a	0.3277	0.2334	0.2835	0.7332	0.6135	0.1197	0.0184
b	0.3277	0.1160	0.1460	0.7709	0.6827	0.0827	−0.0306

Examples 7 and 8. In this environment, constraint (11) can be written as $u_T(t+1) = u_T(0)$. Again, the solution is stationary, with the minor detail that the solution for date zero variables is slightly different. The first order conditions can be reduced to a nonlinear system of 25 equations and 25 variables, which can be solved using Newton's method.

In example 7 the economy starts with low productivity and fiscal deficit. After four years there is a positive technological shock and a decrease in the fiscal deficit. The behavior of fiscal variables and technology is given by

$$(g_t^N, \tau_t, \theta_t^N, \theta_t^T) = \begin{cases} (0.31, 0.10, 0.930, 0.930) & \text{if } t < 60 \\ (0.30, 0.22, 1.015, 1.015) & \text{if } t \geq 60 \ . \end{cases}$$

Although τ has more than doubled, that does not mean the fiscal tightening is large. Recall that the fiscal revenue is given by labor income taxes plus profits. The former is always between 10% and 25% of the total government revenue. The total fiscal revenue increases by less than 16%.

The optimal allocation in example 7 is displayed below.

TABLE 4

EXAMPLE 7 RAMSEY ALLOCATION

t	c_t^T	c_{1t}^N	c_{2t}^N	l_t	l_t^N	l_t^T	x_t
$t = 0$	0.2513	0.0458	0.0781	0.3689	0.2981	0.0707	0.0095
$1 \leq t \leq 60$	0.2514	0.0460	0.0781	0.3690	0.2984	0.0707	0.0094
$t > 60$	0.2707	0.0543	0.0828	0.3253	0.2626	0.0627	-0.0021

It is clear that x decreases. Both l^N and l^T fall, while both θ^N and θ^T increase. Therefore, real wage increase in both sectors. The quantification of e, c, and y is described in part 7.3. The real exchange rate appreciates 1.5%. The intertemporal comparison of consumption and output can be done with three different values for e: the one prior to stabilization, the one prevailing after the stabilization or the current one. Regardless of the real exchange rate used, both c and y increase. Addopting the e that prevails before the stabilization, output, consumption and fiscal revenue grow, respectively, 1.3%, 9.1% and 15.7%, while government expenditures falls 3.2%.

Concerning example 8, fiscal variables and technological shocks follow

$$(g_t^N, \tau_t, \theta_t^N, \theta_t^T) = \begin{cases} (0.26, 0.28, 0.1.05, 1.05) & \text{if } t < 60 \\ (0.28, 0.10, 0.93, 0.93) & \text{if } t \geq 60 \end{cases}.$$

The Ramsey allocation is presented next.

TABLE 5

EXAMPLE 8 RAMSEY ALLOCATION

t	c_t^T	c_{1t}^N	c_{2t}^N	l_t	l_t^N	l_t^T	x_t
$t = 0$	0.2860	0.0585	0.0895	0.2813	0.2230	0.0582	-0.0178
$1 \leq t \leq 60$	0.2861	0.0590	0.0894	0.2815	0.2233	0.0582	-0.0180
$t > 60$	0.2588	0.0466	0.0838	0.3448	0.2729	0.0719	0.0040

The real exchange rate depreciates 3.64%. Consumption and output falls, respectively, 10.8% and 0.1% when evaluated with before crisis real exchange rate, while government expenditures grows 7.7% and the fiscal revenue falls 18.7%.

References

[1] Backus, D.; Kehoe, P. and Kydland, F. (1995). "International Business Cycles: Theory and Evidence." In *Frontier of Business Cycle Research*, ed. Cooley, T., pp. 331-356. New Jersey, Princeton University Press.

[2] Burden, R. and Faires, J. (1997). *Numerical Analysis*. Sixth edition, Pacific Grove, Brooks/Cole Publishing Company.

[3] Chari, V. and Kehoe, P. (1999). "Optimal Fiscal and Monetary Policy." National Bureau of Economic Research, working paper 6891.

[4] Chari, V. and Kehoe, P. (1990)."Sustainable Plans." *Journal of Political Economy* 98, 783-802.

[5] Correia, I; Neves, J. and Rebelo, S. (1995). "Business Cycle in a Small Open Economy." *European Economic Review* 39, 1089-1113.

[6] Frankel, J. and Rose, A. (1996). "Currency Crashes in Emerging Markets: An Empirical Treatment." *Journal of International Economics* 41, 351-366.

[7] Giavazzi, F. and Pagano M. (1990). "Confidence Crises and Public Debt Management." In *Public Debt Management: Theory and History*, ed. Dornbusch, R. and Draghi, M. pp 125-143. Cambridge, Cambridge University Press.

[8] Kiguel, M. and Liviatan, N. (1992). "The Business Cycle Associated with Exchange Rate-Based Stabilization." *The World Bank Economic Review* 6, 279-305.

[9] Klein, M. and Marion, N. (1997). "Explaining the Duration of Exchange Rate Pegs." *Journal of Development Economics* 54, 387-404.

[10] Krugman, P. (1979). “A Model of Balance-of-Payments Crises.” *Journal of Money, Credit and Banking* 11, 311-325.

[11] Kydland, F. and Prescott, E. (1982). “Time to Build and Aggregate Fluctuations.” *Econometrica* 50, 1345-1370.

[12] Lucas, R. and Stokey, N. (1983). “Optimal Fiscal and Monetary Policy in an Economy without Capital.” *Journal of Monetary Economics* 12, 55-93.

[13] Mendoza, E. and Uribe, M. (1997). “The Syndrome of Exchange-Rate-Based Stabilizations and the Uncertain Duration of the Currency Pegs.” Unpublished manuscript.

[14] Milesi-Ferretti, G. and Razin, A. (1998). “Currency Account Reversals and Currency Crises: Empirical Regularities.” National Bureau of Economic Research, working paper 6620.

[15] Nicolini, J. (1998). “More on the Time Consistency of the Monetary Policy.” *Journal of Monetary Economics* 41, 333-350.

[16] Obstfeld, M. (1994). “The Logic of Currency Crises.” *Cahiers Économiques et Monetaires* 43, 189-213.

[17] Obstfeld, M. and Rogoff, K. (1995). “Exchange Rate Dynamics Redux.” *Journal of Political Economy* 102, 624-660.

[18] Rebelo, S. (1997). “What Happens when Countries Peg Their Exchange Rates?”. National Bureau of Economic Research, working paper 6168.

[19] Rebelo, S. and Végh, C. (1995). “Real Effects of Exchange Rate Based Stabilization: An Analysis of Competing Theories.” National Bureau of Economic Research, working paper 5197.

[20] Rebelo, S. and Végh, C. (2001). “When Is it Optimal to Abandon a Fixed Exchange Rate.” Unpublished manuscript.

[21] Végh, C. (1992). “Stopping High Inflation: An Analytical Overview.” *IMF Staff Papers* 39, 626-695.