A NEW PROOF OF THE FIRST WELFARE THEOREM*

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Abstract: The First Welfare Theorem is usually proved by contradiction. In this note we provide a proof by contraposition of this classic theorem.
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Resumo: A demonstração do Primeiro Teorema do Bem-estar usualmente utiliza um argumento de contradição. Fornece-se nesta nota uma prova por contraposicão para esse clássico teorema.
Palavras-Chave: Primeiro Teorema do Bem-Estar, prova por contraposição.

I – Introduction

A central result in general equilibrium theory is the First Welfare Theorem. This theorem shows that, under a relatively small set of assumptions, every competitive equilibrium allocation is Pareto efficient.

The most popular (and in fact the only widely known) way of establishing the First Welfare Theorem is to carry out a proof by contradiction. This proof was first presented by Arrow (1951). With few variations, this is the approach usually found in most textbooks. Some examples are Aliprantis, Brown and Burkinshaw (1990); Mas-Collel, Green and Whinston (1995) and Takayama (1994).

There are some alternative proofs for the First Welfare Theorem. Debreu (1954) showed that if an allocation is feasible, then it cannot Pareto dominate a competitive equilibrium allocation. Debreu (1983) showed that any allocation that makes some agent better-off without harming another (when compared to a competitive equilibrium

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allocation) is not feasible. Ellickson (1993) provided another proof (a direct one) to that classic theorem.

In this note we establish the First Welfare Theorem by means of contraposition reasoning. We show that if an allocation is not Pareto efficient, then this allocation is not a competitive equilibrium allocation.

This new proof establishes a direct connection between the concepts of competitive equilibrium and Pareto efficiency. It clearly shows that no price system can support as a competitive equilibrium an allocation that is not Pareto efficient. This is exactly the same argument used to convince an undergraduate student that an allocation that does not lay on the contract curve of an Edgeworth's box is not a competitive equilibrium allocation.

This note is organized as follows. For simplicity, the new approach to prove the First Welfare Theorem is initially used in a pure exchange economy in Section II. The argument is generalized to an economy with production in Section III.

II – A Pure Exchange Economy

There exist a set $\mathbf{I} = \{1,2,...,I\}$ of consumers and a set $\mathbf{L} = \{1,2,...,L\}$ of commodities. The commodity space is $\Re^L_+$. Each consumer $i \in \mathbf{I}$ has a preference relation $\succeq_i$ on her consumption set $\mathbf{X}_i \subseteq \Re^L_+$ and an initial endowment $\varpi_i \in \mathbf{X}_i$. As usual, $x_i \succ_i \widetilde{x}_i$ means that $x_i \succeq_i \widetilde{x}_i$ and $\neg(\widetilde{x}_i \succeq_i x_i)$. An allocation is a vector $x \in \Re^{IL}_+$. It can be written as $x = (x_1, x_2,\ldots,x_I)$, where each $x_i \in \mathbf{X}_i$. A price system is any vector $p \in \Re^L_+$. Given a price system p, the budget set of consumer i is the set $\mathbf{B}_i(p) = \{x_i \in \mathbf{X}_i : p\cdot(x_i - \varpi_i) \leq 0\}$. The next five definitions spell out the remaining introductory formalities.

Definition 1 A preference relation $\succeq_i$ is *locally non-satiable* if for every $\bar{x}_i \in \mathbf{X}_i$ and all $\delta > 0$ there exists $\tilde{x}_i \in \{x_i \in \mathbf{X}_i : \|x_i - \bar{x}_i\| < \delta\}$ satisfying $\tilde{x}_i \succ_i \bar{x}_i$.

Definition 2 A bundle $x_i \in \mathbf{X}_i$ is a *maximal element* for $\succeq_i$ in a set $\mathbf{V}_i \subseteq \mathbf{X}_i$ if $x_i \succ_i \tilde{x}_i$ for all $\tilde{x}_i$ in $\mathbf{V}_i$.

Definition 3 An allocation x is *feasible* if $x_i \in \mathbf{X}_i$ for all i and $\sum_{i=1}^{I} x_i = \sum_{i=1}^{I} \varpi_i$.

Definition 4 An allocation x is *Pareto efficient* if it is feasible and there is no feasible allocation $\tilde{x}$ that satisfies $\tilde{x}_i \succeq_i x_i$ for all i and $\tilde{x}_i \succ_i x_i$ for some i.

Definition 5 A *competitive equilibrium* is a vector (p,x) that satisfies: (i) x is feasible and (ii) for each i, x_i is a maximal element for $\succeq_i$ in $\mathbf{B}_i(p)$.

Theorem 1 Suppose that each $\succeq_i$ is locally non-satiable. If (p,x) is a competitive equilibrium, then x is Pareto efficient.

Proof It is enough to show that if an allocation x is not Pareto efficient, then there is no price system p such that (p,x) is a competitive equilibrium. Take any x that is not Pareto efficient and any price system p. Either (i) x is not feasible or (ii) x is feasible and there exists another feasible allocation $\bar{x}$ that satisfies $\bar{x}_i \succeq_i x_i$ for all i and $\bar{x}_i \succ_i x_i$ for some i. If (i) is true there is nothing to show.

Consider the situation in which (ii) holds. Define the sets $\mathbf{I}_{\succ_i} = \{i \in \mathbf{I}: \bar{x}_i \succ_i x_i\}$, $\mathbf{I}_{>} = \{i \in \mathbf{I}: p\cdot(x_i - \varpi_i) > 0\}$, $\mathbf{I}_{<} = \{i \in \mathbf{I}: p\cdot(x_i - \varpi_i) < 0\}$, $\mathbf{I}_{=} = \{i \in \mathbf{I}: p\cdot(x_i - \varpi_i) = 0\}$, and $\mathbf{I}_{\leq} = \mathbf{I}_{<} \cup \mathbf{I}_{=}$. Note that the family $\{\mathbf{I}_{>}, \mathbf{I}_{=}, \mathbf{I}_{<}\}$ constitutes a partition of $\mathbf{I}$. Moreover, $\mathbf{I}_{\succ_i} \neq \varnothing$.

If $\mathbf{I}_{\succ_i} \cap \mathbf{I}_{\leq} = \varnothing$, then $\mathbf{I}_{\succ_i} \cap \mathbf{I}_{>} \neq \varnothing$. Hence, $\mathbf{I}_{>} \neq \varnothing$ and

$$\sum_{i \in \mathbf{I}_>} p \cdot (\bar{x}_i - \varpi_i) > 0 \ .$$

On the other hand, $\bar{x}$ is feasible. We then have

$$\sum_{i=1}^{I} p \cdot (\bar{x}_i - \varpi_i) = 0 \Rightarrow \sum_{i \in \mathbf{I}_<} p \cdot (\bar{x}_i - \varpi_i) = -\sum_{i \in \mathbf{I}_>} p \cdot (\bar{x}_i - \varpi_i) \ .$$

We conclude that

$$\sum_{i \in \mathbf{I}_<} p \cdot (\bar{x}_i - \varpi_i) < 0 \ .$$

Thus, $\mathbf{I}_<$ is not empty. Since $\succeq_i$ is locally non-satiable, there is an agent $i \in \mathbf{I}_<$ and a bundle $\tilde{x}_i \in \mathbf{B}_i(p)$ satisfying $\tilde{x}_i >_i \bar{x}_i \succeq_i x_i$. Therefore, x_i is not a maximal element for $\succeq_i$ in $\mathbf{B}_i(p)$, from which follows that (p,x) is not a competitive equilibrium.

We finish the proof by considering the case in which (ii) holds and $\mathbf{I}_{\succ_i} \cap \mathbf{I}_{\leq} \neq \varnothing$. Clearly, in such a context, there is some $i \in \mathbf{I}$ for which x_i is not a maximal element for $\succeq_i$ in $\mathbf{B}_i(p)$. Hence, (p,x) cannot be a competitive equilibrium.

III – A Production Economy

The environment builds on the one described in Section II. There exists a set $\mathbf{I} = \{1,2,\ldots,I\}$ of consumers, a set $\mathbf{L} = \{1,2,\ldots,L\}$ of commodities and a set $\mathbf{J} = \{1,2,\ldots,J\}$ of firms. The commodity space is $\Re_+^L$. Each firm $j \in \mathbf{J}$ has a production set $\mathbf{Y}_j \subseteq \Re_+^L$. Each consumer $i \in \mathbf{I}$ has a preference relation $\succeq_i$ on her consumption set $\mathbf{X}_i \subseteq \Re_+^L$ and an initial endowment $\varpi_i \in \mathbf{X}_i$. An allocation is a vector $(x,y) \in \Re_+^{(I+J)L}$, where $x \in \Re_+^{IL}$ and $y \in \Re_+^{JL}$. It can be written as $(x,y) = (x_1, x_2,\ldots,x_I, y_1, y_2,\ldots, y_J)$, where $x_i \in \mathbf{X}_i$ and $y_j \in \mathbf{Y}_j$. A price system is any vector $p \in \Re_+^L$. The matrix $\Theta = [\theta_{ij}]_{I \times J}$, where $\theta_{ij} \geq 0$, describes the share of firm j's profit that is owned by consumer i. Of course, $\sum_{i=1}^{I} \theta_{ij} = 1$ for every j. The budget set of consumer i is

$$\mathbf{B}_i(p, y) = \left\{x_i \in \mathbf{X}_i : p \cdot (x_i - \varpi_i) - \sum_{j=1}^{J} \theta_{ij} p \cdot y_j \leq 0\right\}.$$

Locally non-satiable preferences and maximal element are defined as in the previous section. The remaining definitions are:

Definition 6 An allocation (x,y) is feasible if $x_i \in \mathbf{X}_i$ for all i, $y_j \in \mathbf{Y}_j$ for all j and $\sum_{i=1}^{I} x_i = \sum_{i=1}^{I} \varpi_i + \sum_{j=1}^{J} y_j$.

Definition 7 An allocation (x,y) is *Pareto efficient* if it is feasible and there is no feasible allocation $(\tilde{x}, \tilde{y})$ that satisfies $\tilde{x}_i \succeq_i x_i$ for all i and $\tilde{x}_i \succ_i x_i$ for some i.

Definition 8 A competitive equilibrium is a vector (p,x,y) that satisfies: (i) (x,y) is feasible; (ii) for each j, $p \cdot y_j \geq p \cdot \tilde{y}_j$ for all $\tilde{y}_j \in \mathbf{Y}_j$; and (iii) for each i, x_i is a maximal element for $\succeq_i$ in $\mathbf{B}_i(p,y)$.

Theorem 2 Suppose that each $\succeq_i$ is locally non-satiable. If (p,x,y) is a competitive equilibrium, then (x,y) is Pareto efficient.

Proof Take an allocation (x,y) that is not Pareto efficient and a price system p. It suffices to show that (p,x,y) is not a competitive equilibrium. Either (i) (x,y) is not feasible or (ii) (x,y) is feasible and there exists another feasible allocation $(\bar{x}, \bar{y})$ that satisfies $\bar{x}_i \succeq_i x_i$ for all i and $\bar{x}_i \succ_i x_i$ for some i. If (i) is true there is nothing to show.

If (ii) holds, then either $p\cdot \bar{y}_j \leq p\cdot y_j$ for all j or that inequality fails for some firm j. Clearly, in this last case (p,x,y) cannot be a competitive equilibrium. Therefore, from now on we assume that $p\cdot \bar{y}_j \leq p\cdot y_j$ for all j and closely follow the proof of Theorem 1.

Let the set $\mathbf{I}_{\succ_i}$ be as defined in the previous proof. We define $\mathbf{I}_{>}$ according to

$$\mathbf{I}_{>} = \left\{ i \in \mathbf{I} : p\cdot(\bar{x}_i - \varpi_i) - \sum_{j=1}^{J} \theta_{ij}\, p\cdot y_j > 0 \right\}.$$

The sets $\mathbf{I}_{=}$, $\mathbf{I}_{<}$ and $\mathbf{I}_{\leq}$ are similarly defined.

As before, if $\mathbf{I}_{\succ_i} \cap \mathbf{I}_{\leq} \neq \varnothing$, then there is some $i \in \mathbf{I}$ for which x_i is not a maximal element for $\succeq_i$ in $\mathbf{B}_i(p,y)$ and (p,x,y) is not a competitive equilibrium. For the case in which $\mathbf{I}_{\succ_i} \cap \mathbf{I}_{\leq} = \varnothing$, we again use the fact that $\mathbf{I}_{>} \neq \varnothing$. Therefore,

$$\sum_{i\in\mathbf{I}_{>}} \left[p\cdot(\bar{x}_i - \varpi_i) - \sum_{j=1}^{J} \theta_{ij}\, p\cdot y_j \right] > 0 \ . \tag{1}$$

On the other hand, the fact that $(\bar{x},\bar{y})$ is feasible implies

$$\sum_{i=1}^{I} p\cdot(\bar{x}_i - \varpi_i) = \sum_{j=1}^{J} p\cdot \bar{y}_j = \sum_{j=1}^{J}\sum_{i=1}^{I} \theta_{ij}\, p\cdot \bar{y}_j \leq \sum_{j=1}^{J}\sum_{i=1}^{I} \theta_{ij}\, p\cdot y_j \Rightarrow$$

$$\sum_{i=1}^{I} p\cdot(\bar{x}_i - \varpi_i) \leq \sum_{j=1}^{J}\sum_{i=1}^{I} \theta_{ij}\, p\cdot y_j \Rightarrow \sum_{i=1}^{I} \left[p\cdot(\bar{x}_i - \varpi_i) - \sum_{j=1}^{J} \theta_{ij}\, p\cdot y_j \right] \leq 0 \Rightarrow$$

$$\sum_{i\in\mathbf{I}_{<}} \left[p\cdot(\bar{x}_i - \varpi_i) - \sum_{j=1}^{J} \theta_{ij}\, p\cdot y_j \right] \leq - \sum_{i\in\mathbf{I}_{>}} \left[p\cdot(\bar{x}_i - \varpi_i) - \sum_{j=1}^{J} \theta_{ij}\, p\cdot y_j \right] .$$

We combine the last inequality with (1) to conclude that

$$\sum_{i\in\mathbf{I}_{<}} \left[p\cdot(\bar{x}_i - \varpi_i) - \sum_{j=1}^{J} \theta_{ij}\, p\cdot y_j \right] < 0 \ .$$

The reasoning adopted in the previous proof establishes that (p,x,y) is not a competitive equilibrium.

References

Aliprantis, C.; Brown, D. and Burkinshaw, O. *Existence and Optimality of Competitive Equilibrium*. Berlin: Springer-Verlag, 1990.

Arrow, K. An extension of the basic theorems of classical Welfare Economics. In *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*. Berkeley: University of California, 1951.

Debreu, G. *Theory of Value: An Axiomatic Analysis of Economic Equilibrium*. New Haven: Yale University Press, 1954.

Debreu, G. *Mathematical Economics: Twenty Papers of Gerard Debreu*. Cambridge: Cambridge University Press, 1983.

Ellickson, B. *Competitive Equilibrium: Theory and Applications*. Cambridge: Cambridge University Press, 1993.

Mas-Collel, A.; Green, J. and Whinston, M. *Microeconomic Theory*. Oxford: Oxford University Press, 1995.

Takayama, A. *Mathematical Economics*. Second Edition. Cambridge: Cambridge University Press, 1994.